Automata, Transducers, and Hidden Markov Models

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CS 421: Natural Language Processing

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Many slides adapted from Jurafsky and Martin (https://web.stanford.edu/~jurafsky/slp3/), Universiteit Utrecht's NLP course (http://www.phil.uu.nl/tst/2012/Slides/SLP_Lecture2.pdf), and Ray Mooney's NLP course (https://www.cs.utexas.edu/~mooney/cs388/).

What are finite state automata?

- **Computational models that can generate regular languages** (such as those specified by a regular expression)
- Also used in other NLP applications that function by **transitioning between finite states**
  - Dialogue systems
  - Morphological parsing
- Singular: Finite State Automaton (FSA)
- Plural: Finite State Automata (FSAs)

Key Components

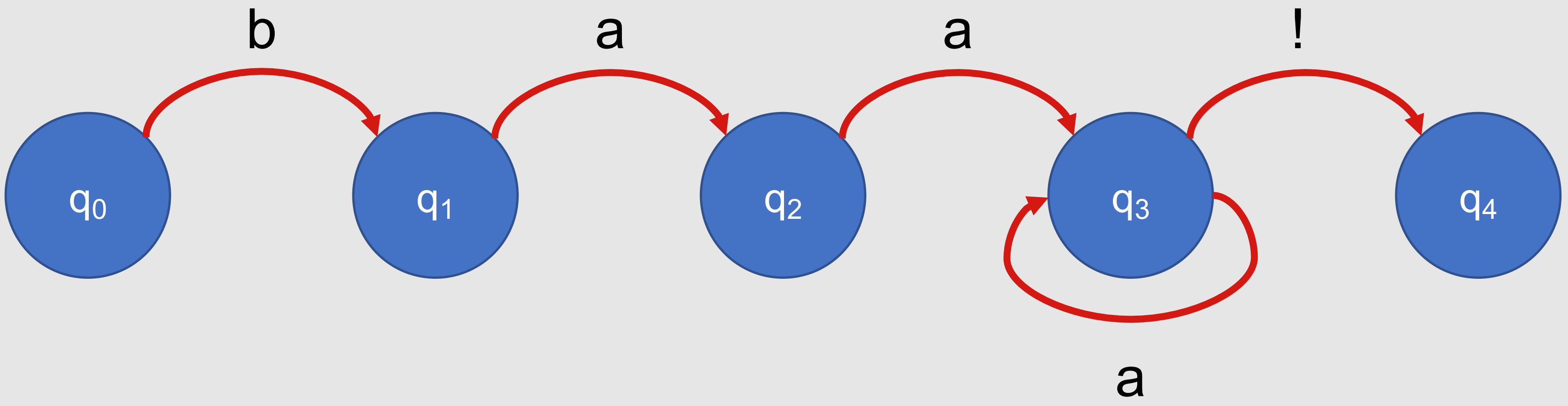
- Finite set of states
  - Start state
  - Final state
- Set of transitions from one state to another

How do FSAs work?

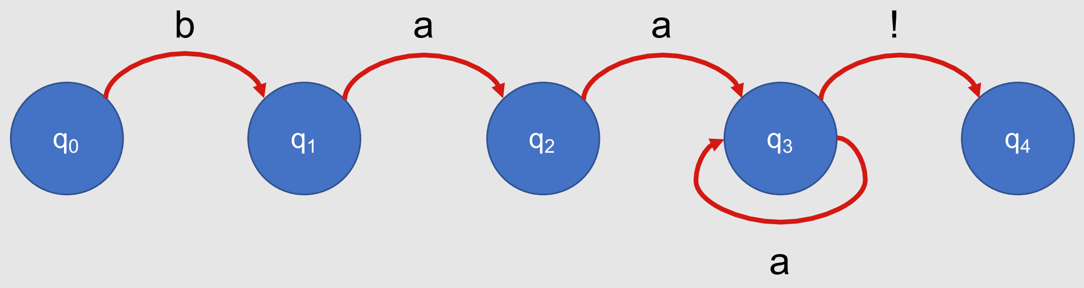
- For a given sequence of items (characters, words, etc.) to match, **begin in the start state**
- **If the next item** in the sequence **matches a state that can be transitioned to** from the current state, **go to that state**
- **Repeat**
  - If no transitions are possible, **stop**
  - If the state you stopped in is a final state, **accept the sequence**

FSAs are often represented graphically.

- Nodes = states
- Arcs = transitions



What do we know about this FSA?

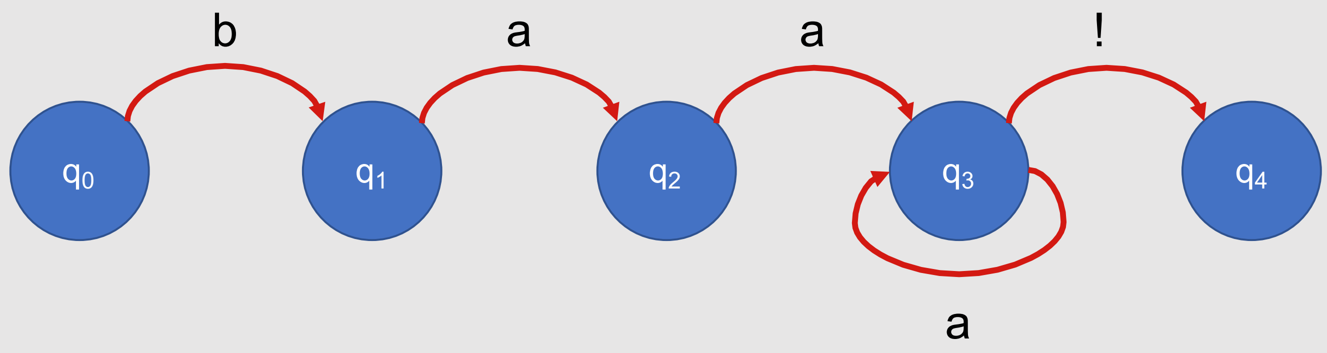


- Five states
  - q_0 is the start state
  - q_4 is the final (accept) state
- Five transitions
- Alphabet = {a, b, !}

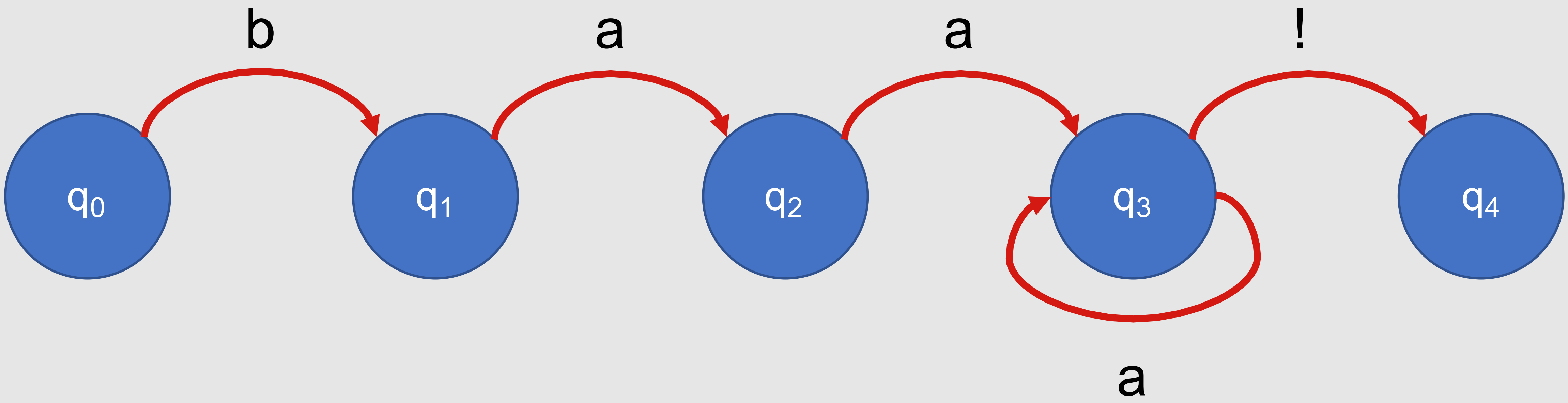
Which strings could this FSA match?

- baa!
- baaaa!
- ba!
- baaaaaaaa!
- baaaa
- baabaa!

- https://www.google.com/search?q=timer



Regex that this FSA matches: baa+!



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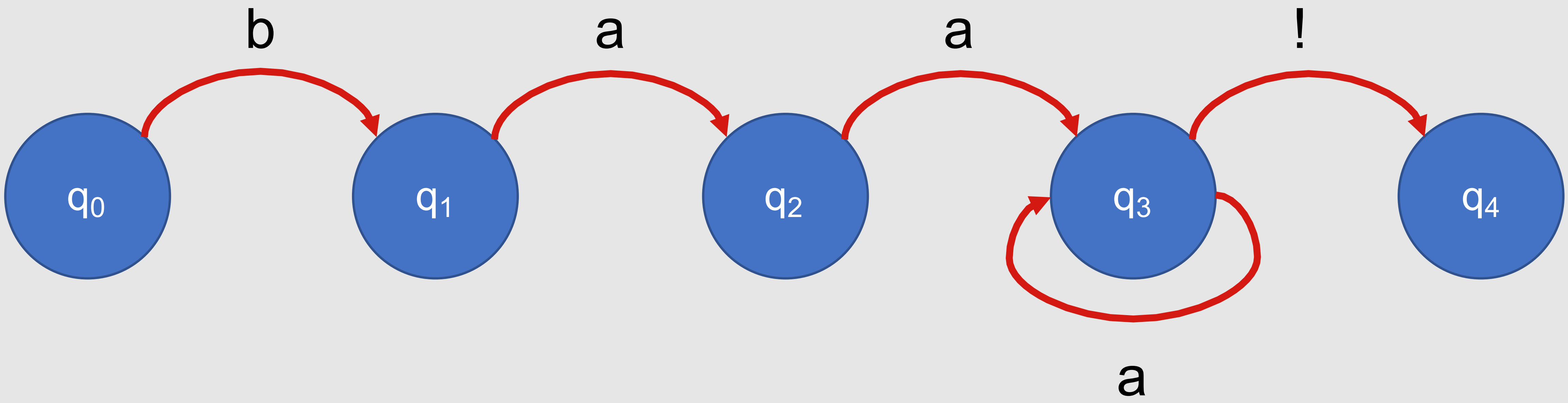
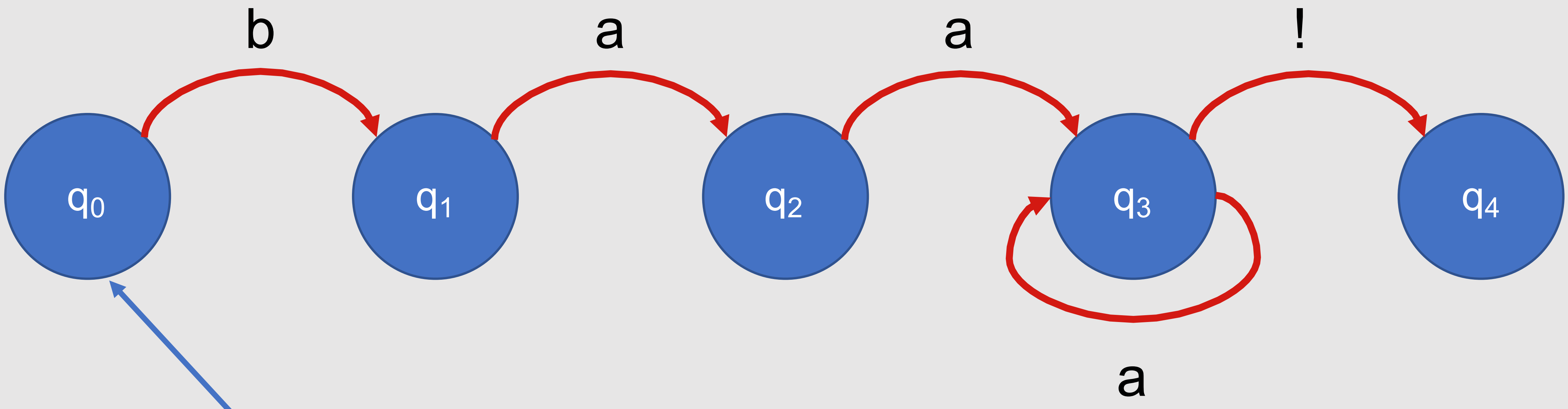


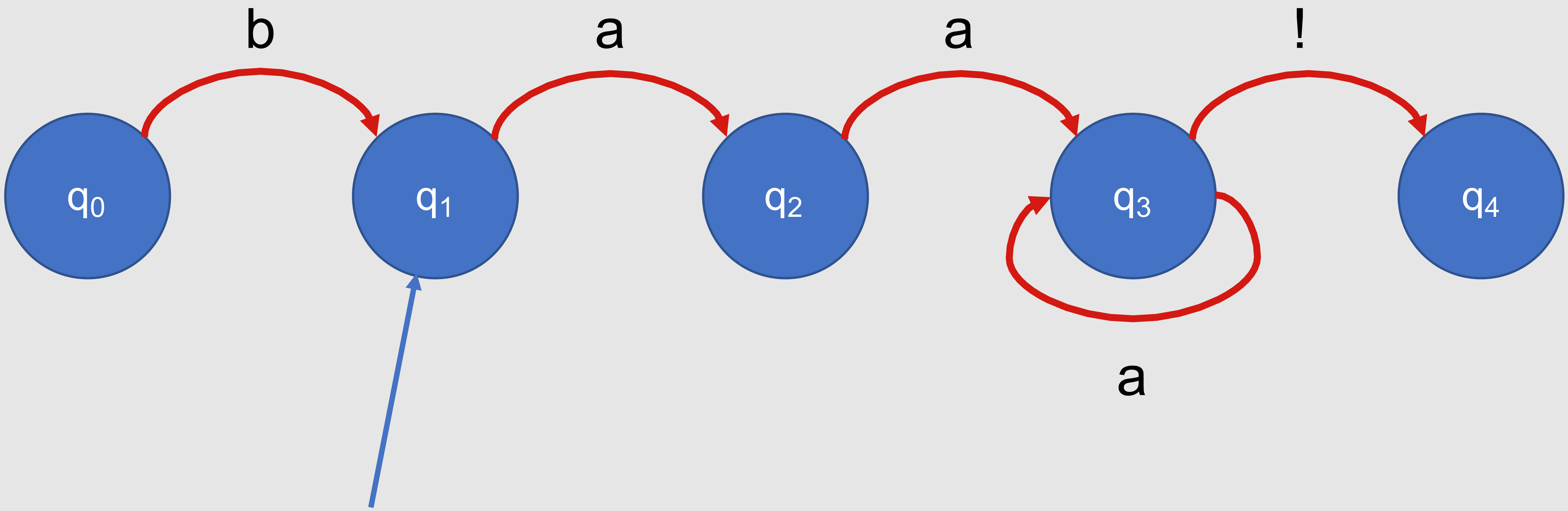
Test String: baa!

Regex that this FSA matches: baa+!



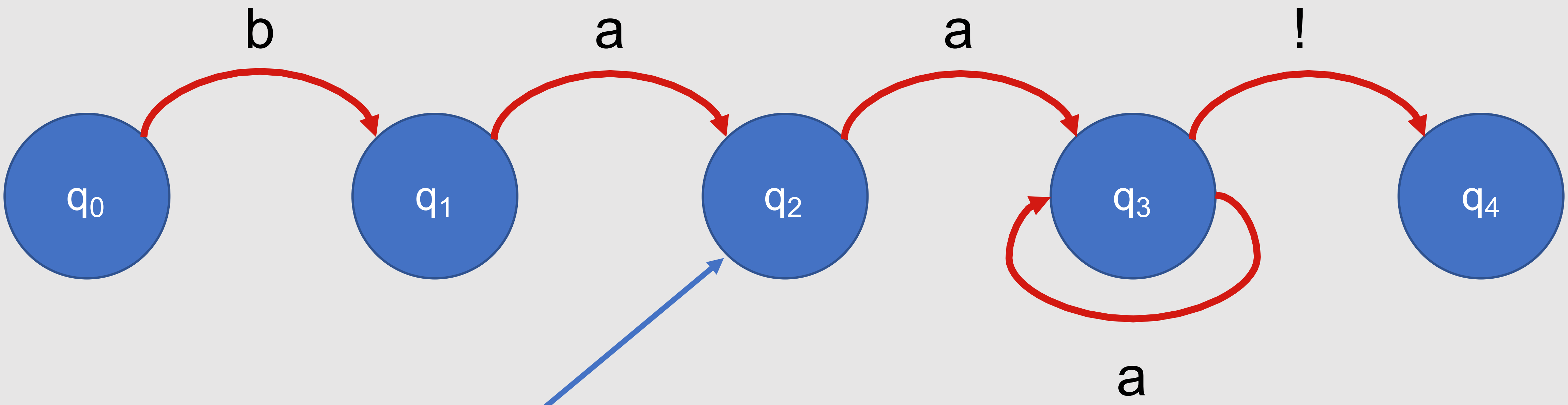
Test String: baa!

Regex that this FSA matches: baa+!



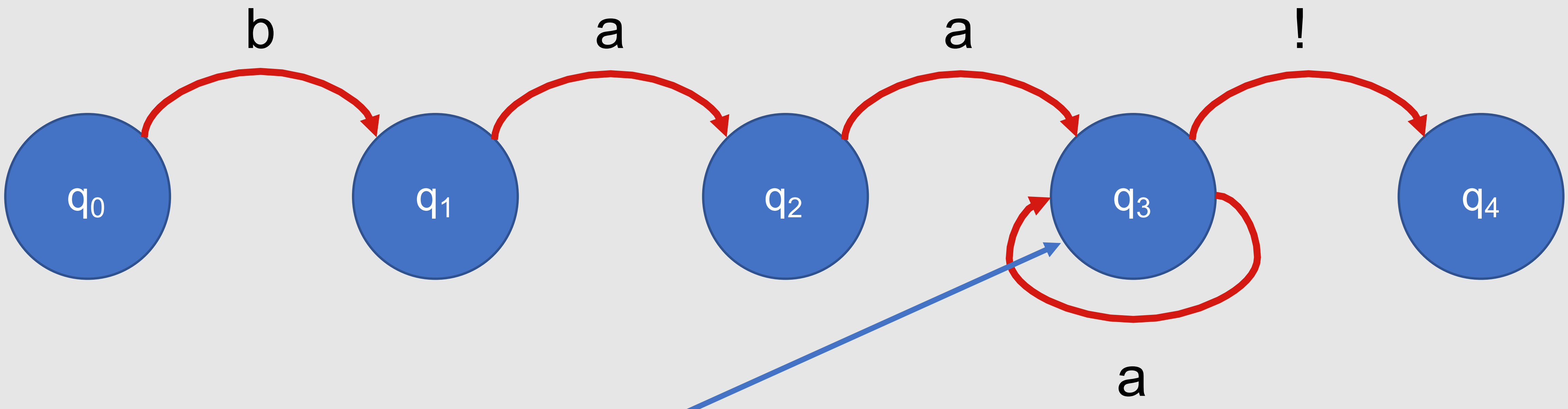
Test String: **b**aa!

Regex that this FSA matches: baa+!



Test String: baa!

Regex that this FSA matches: baa+!



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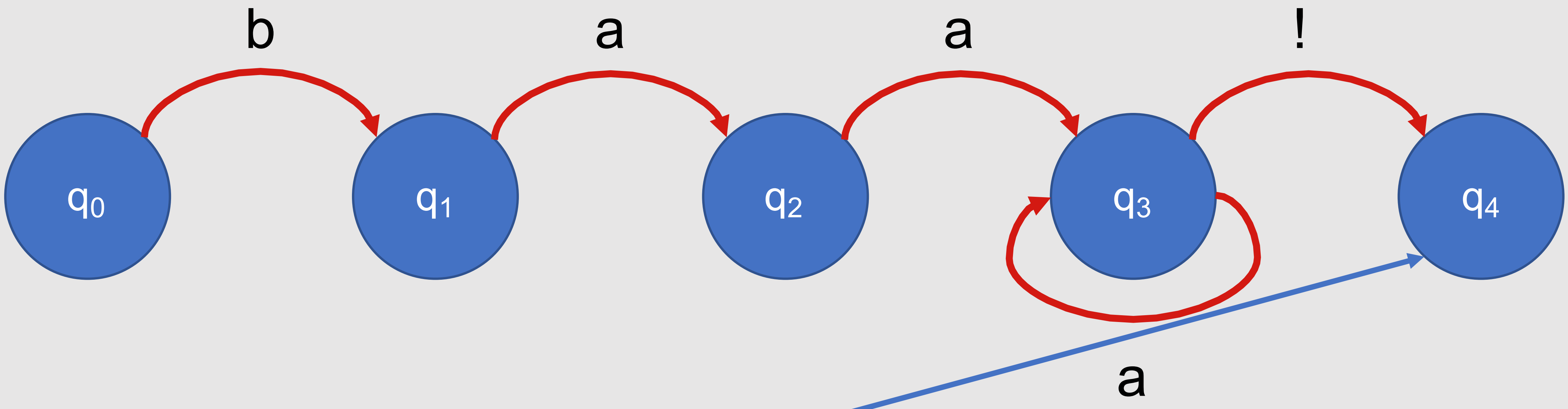
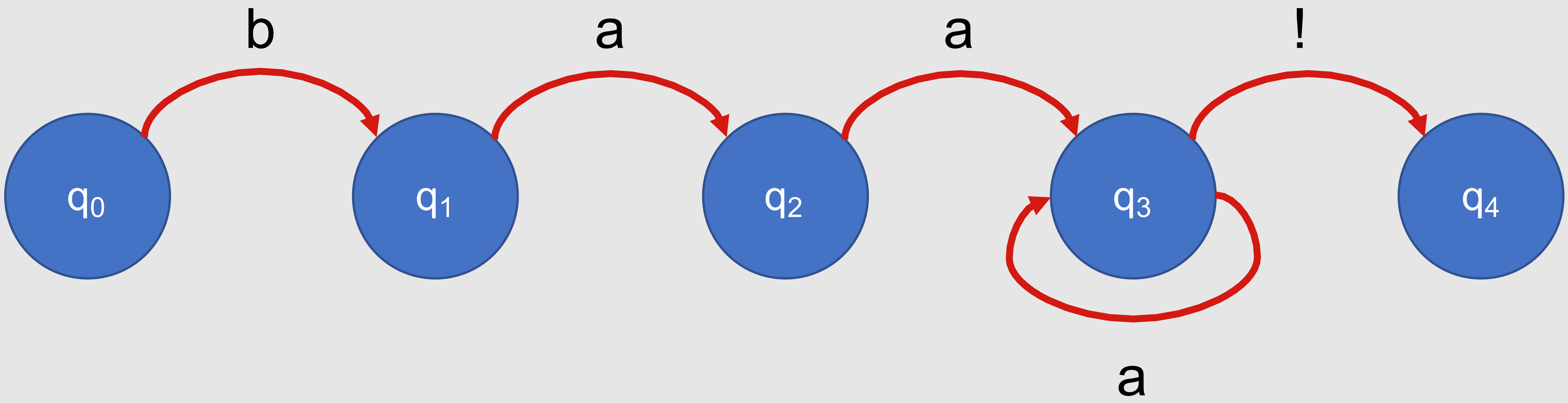
Regex that this FSA matches: baa+!



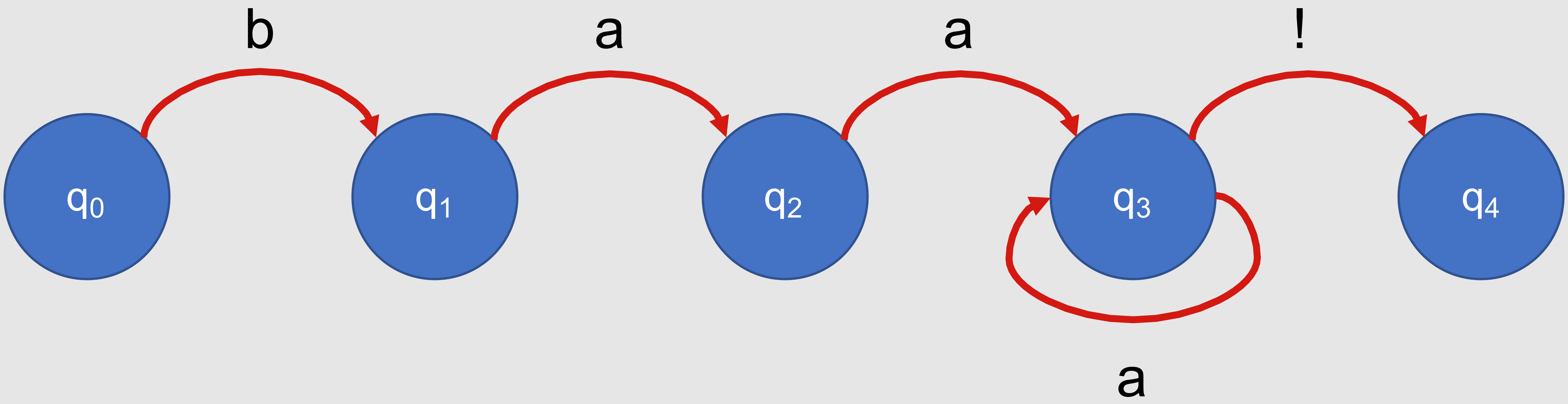
Test String: baa!

Regex that this FSA matches: baa+!



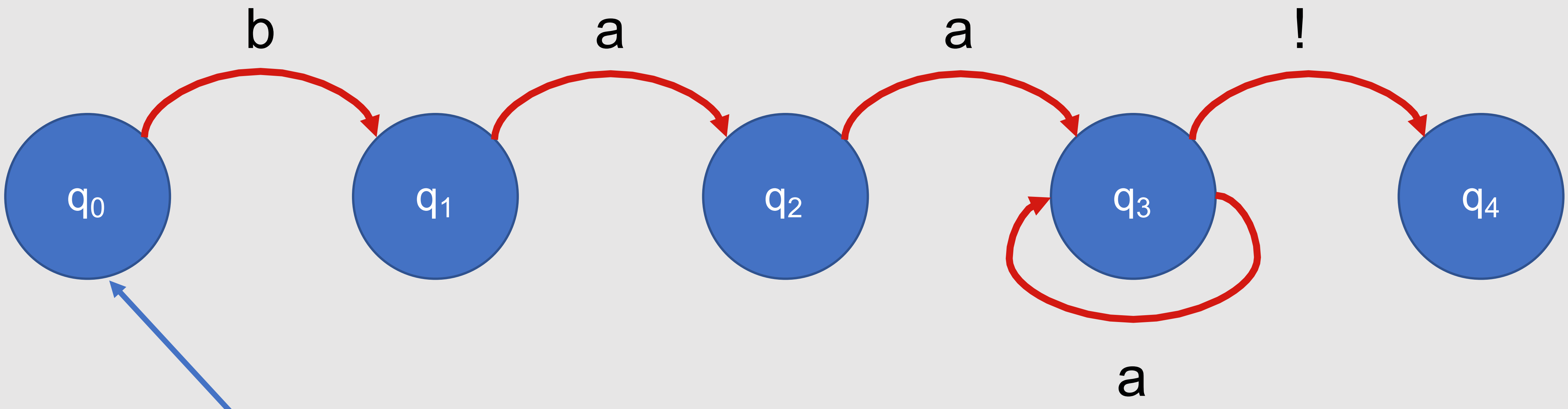
Test String: baa! ☺

Regex that this FSA matches: baa+!



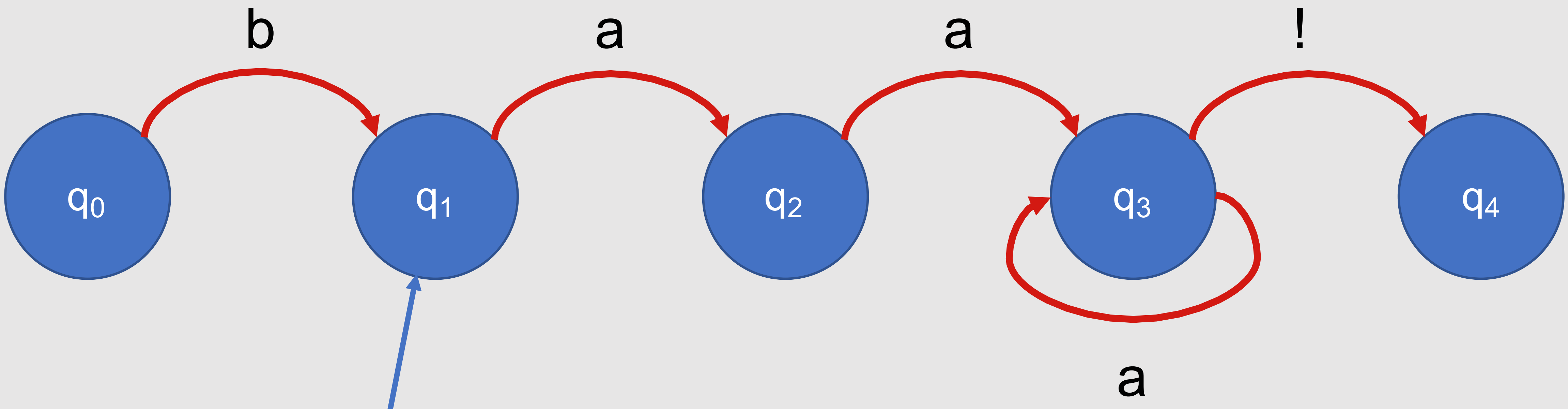
Test String: baabaa!

Regex that this FSA matches: baa+!



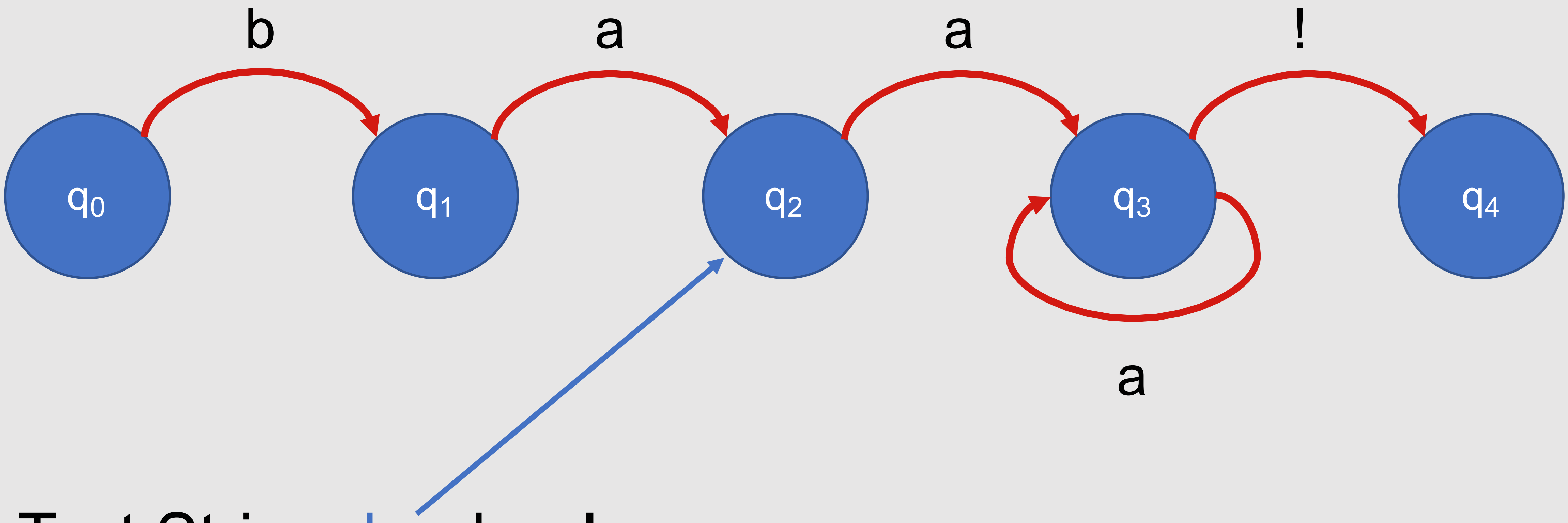
Test String: baabaa!

Regex that this FSA matches: baa+!



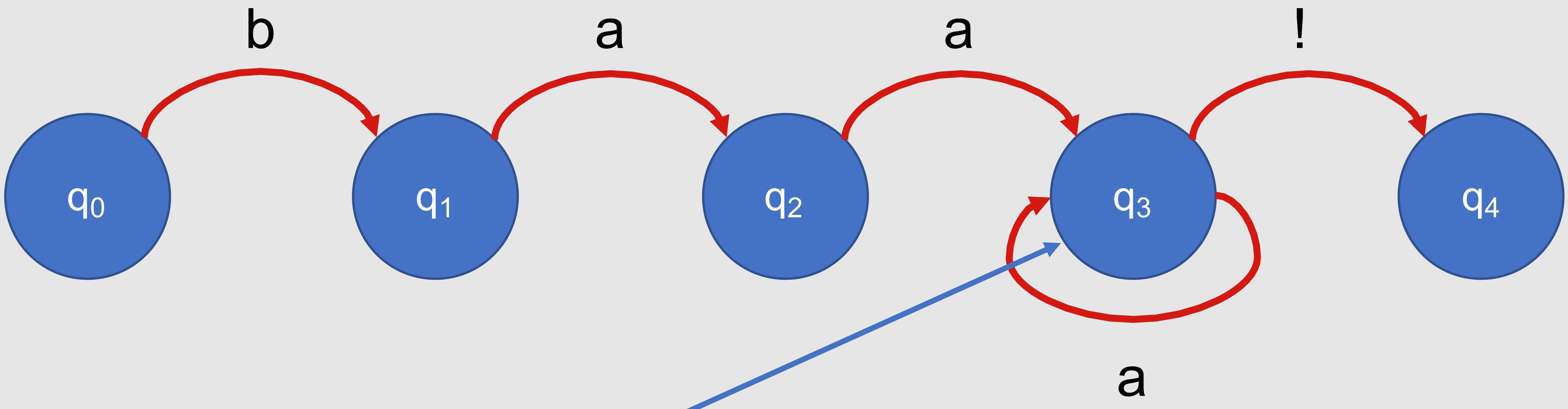
Test String: baabaa!

Regex that this FSA matches: baa+!



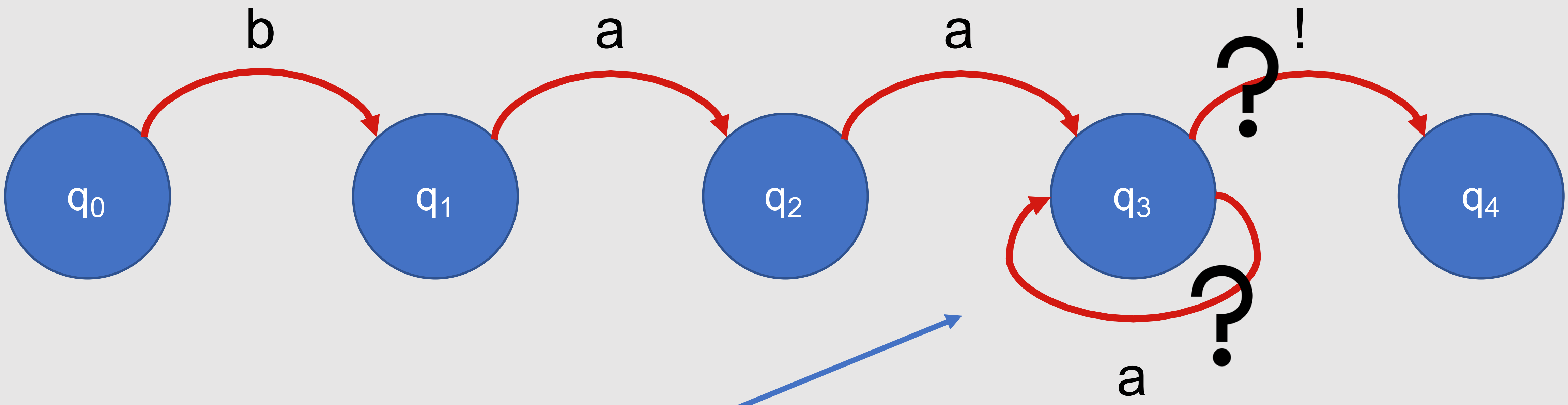
Test String: baabaa!

Regex that this FSA matches: baa+!



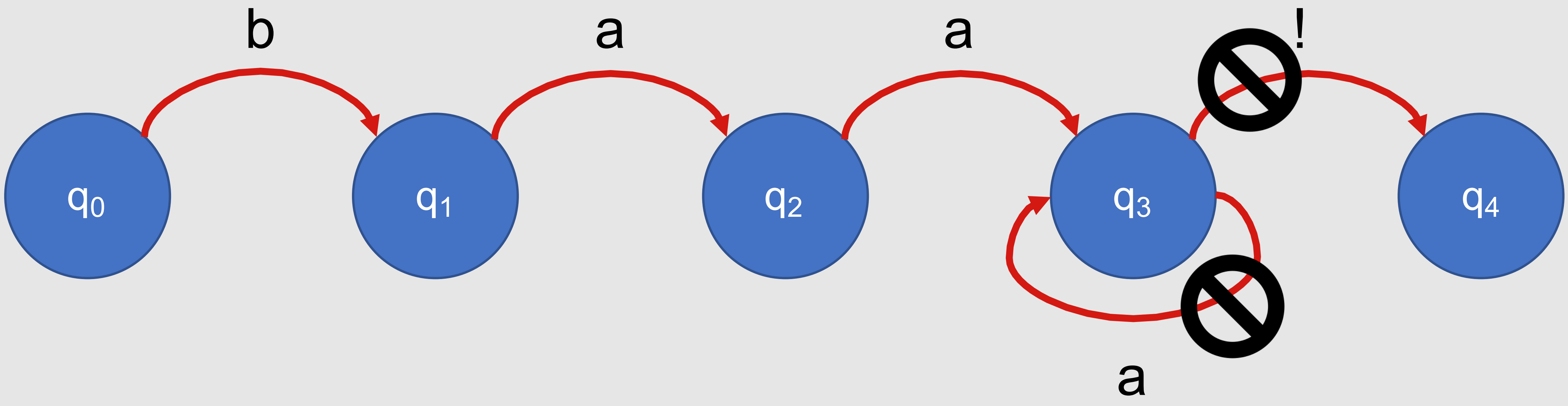
Test String: baabaa!

Regex that this FSA matches: baa+!



Test String: baabaa! ☹

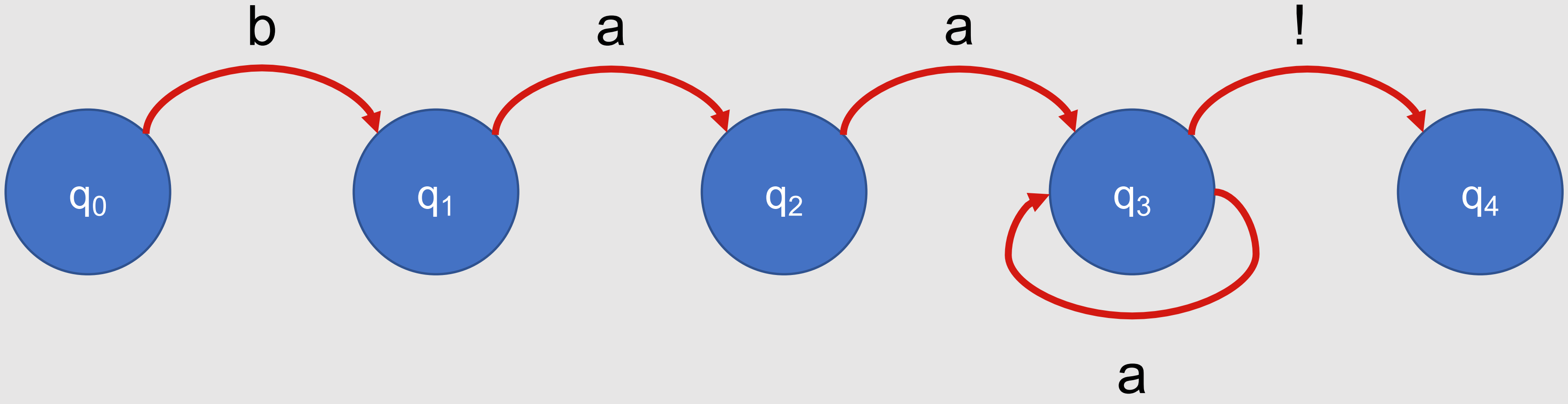
Regex that this FSA matches: baa+!



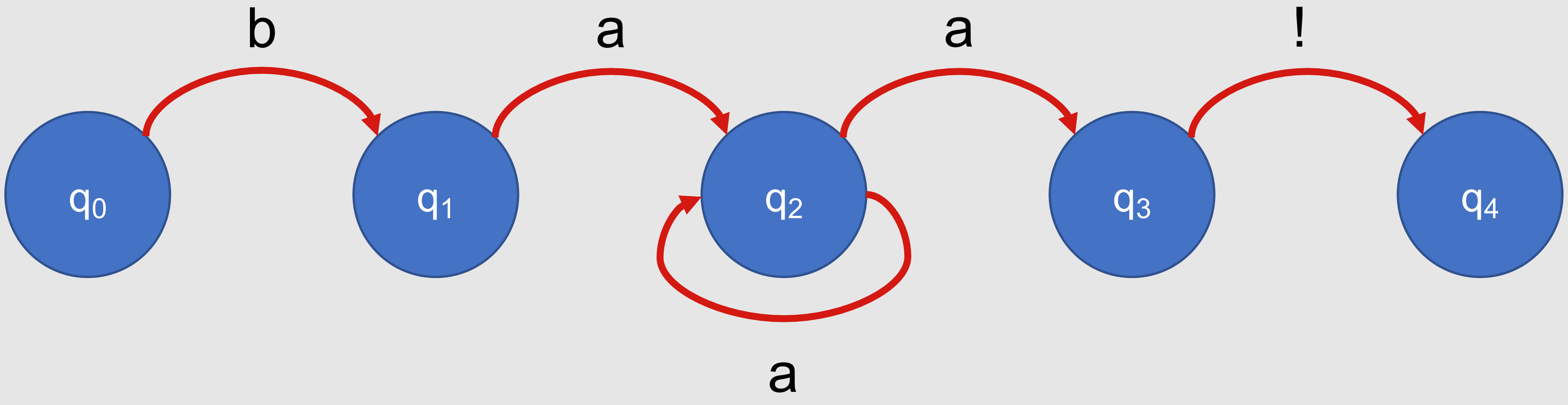
Test String: baabaa! ☹

Note: More than one FSA can correspond to the same regular language!

Test String:
baaa!



Test String:
baaa!



Formal Definition

- A finite state automaton can be specified by enumerating the following properties:
  - The set of states, ***Q***
  - A finite alphabet, ***Σ***
  - A start state, q_0
  - A set of accept/final states, $F \subseteq Q$
  - A transition function or transition matrix between states, $\delta(q,i)$
- $\delta(q,i)$: Given a state $q \in Q$ and input $i \in \Sigma$, **$\delta(q,i)$ returns a new state $q' \in Q$**.

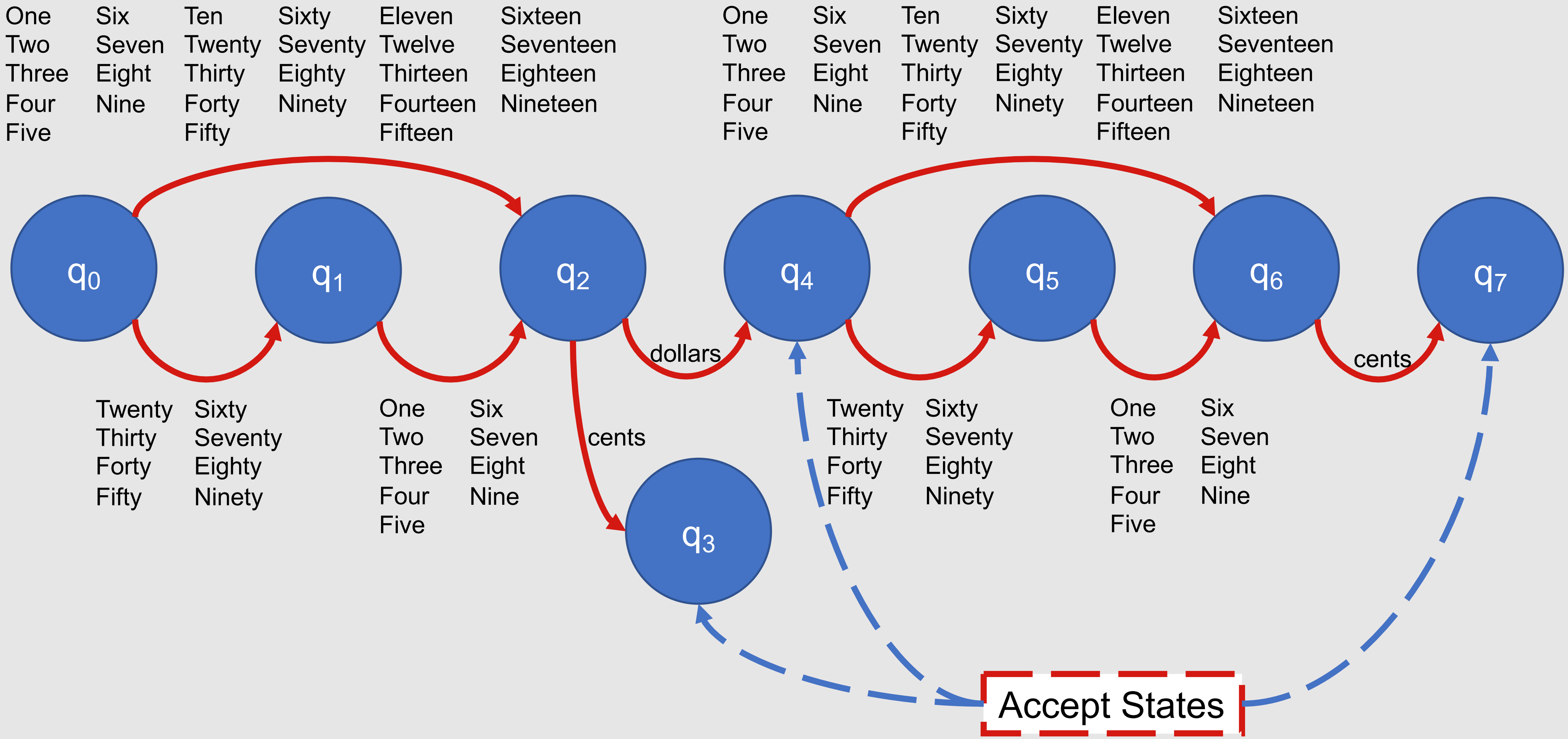
Alphabets

In the previous definition, alphabet does not necessarily mean [a-zA-Z]!

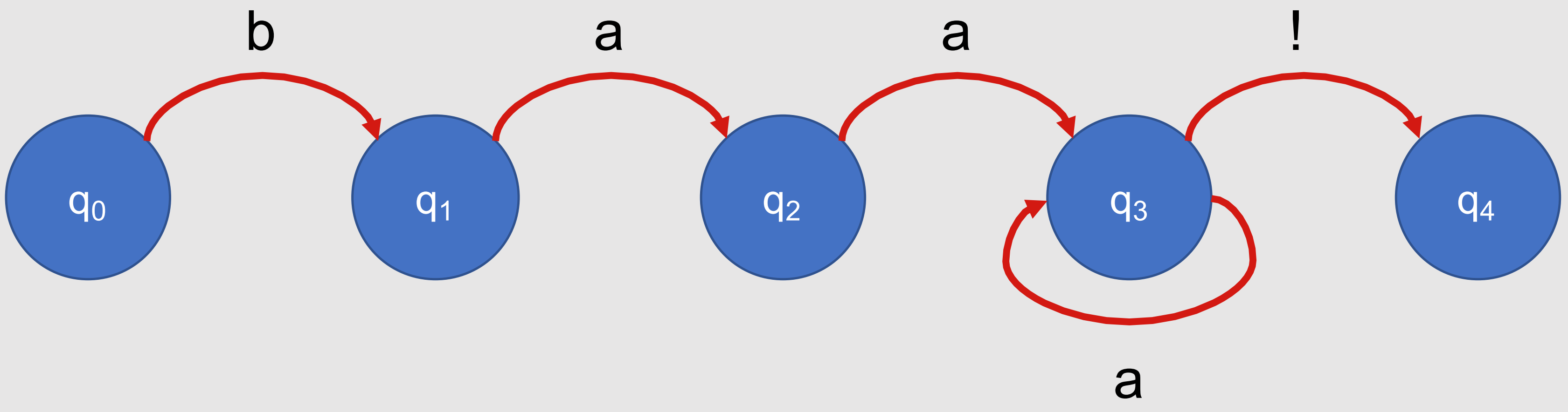
Alphabet = finite set of possible input symbols

An alphabet can be a subset of letters (e.g., {a, b}), a combination of letters and other characters (e.g., {a, b, !}), a subset of words (e.g., {lamb, sheep, baa!}), etc.

Example: FSA for Dollar Amounts



State transitions in FSAs can be represented using tables.



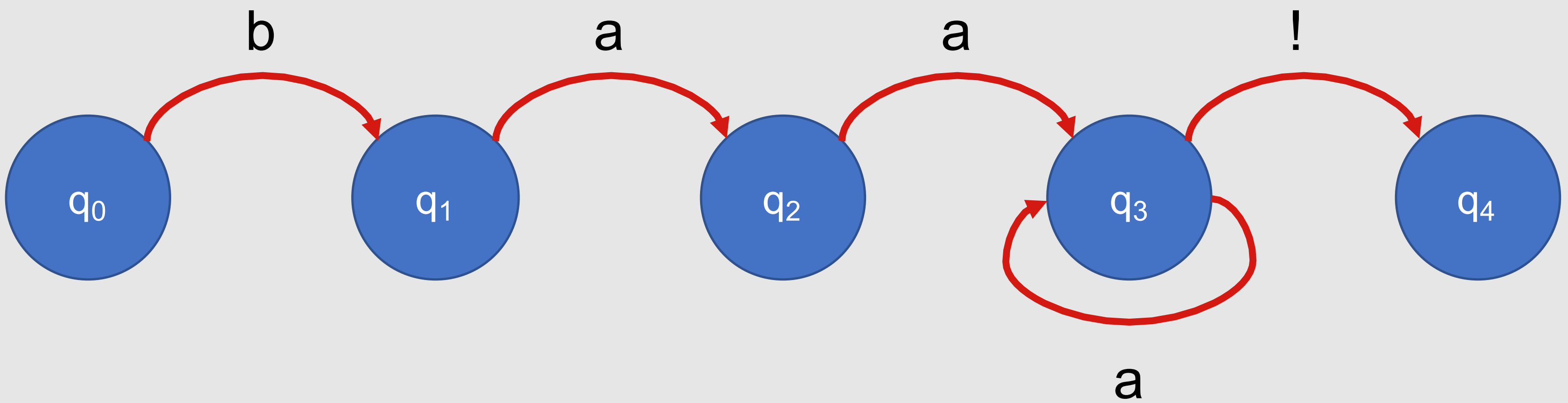
Next Item in Sequence

Currently in State

	b	a	!	<end>
q_0	q_1			
q_1				
q_2				
q_3				
q_4				

Go to State

State transitions in FSAs can be represented using tables.



Next Item in Sequence

Currently in State

	b	a	!	<end>
q_0	q_1	☹	☹	☹
q_1				
q_2				
q_3				
q_4				

Go to State

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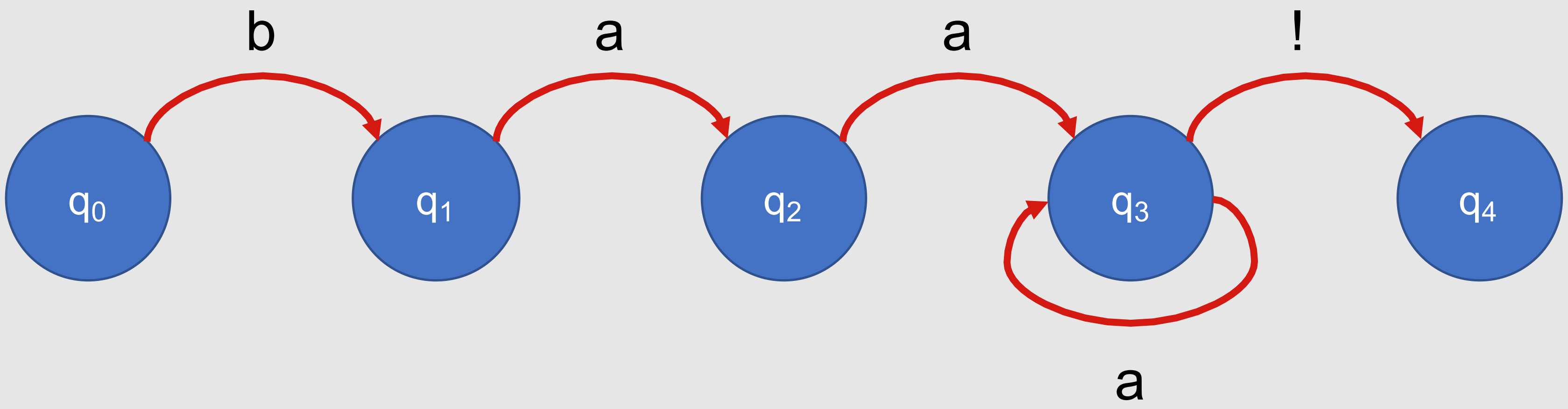

Next Item in Sequence

Currently in State

	b	a	!	<end>
q_0	q_1	☹	☹	☹
q_1	☹	q_2		
q_2				
q_3				
q_4				

Go to State

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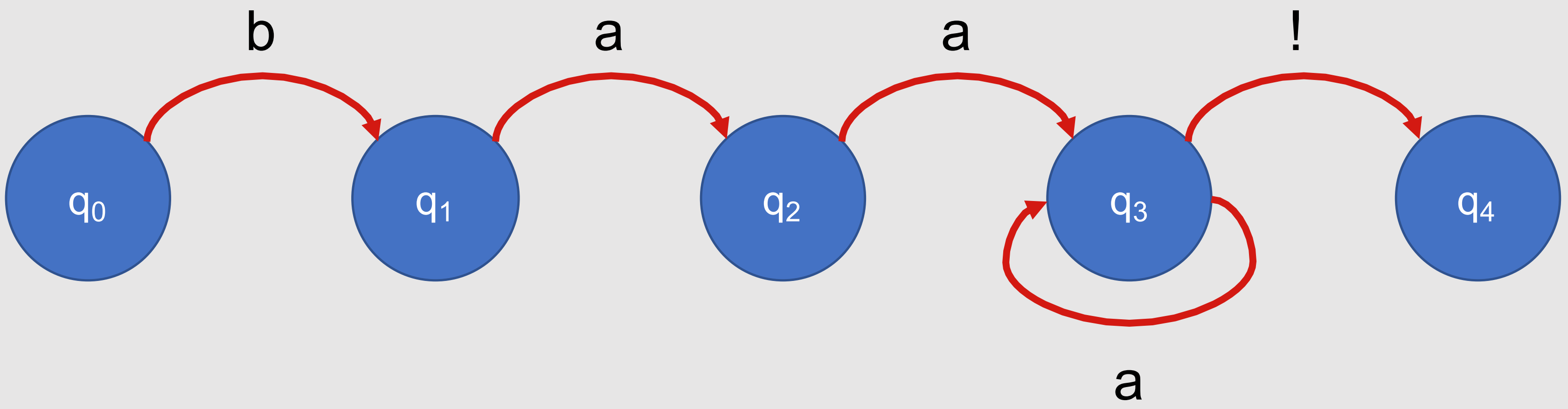

Next Item in Sequence

Currently in State

	b	a	!	<end>
q_0	q_1	☹	☹	☹
q_1	☹	q_2	☹	☹
q_2	☹	q_3		
q_3				
q_4				

Go to State

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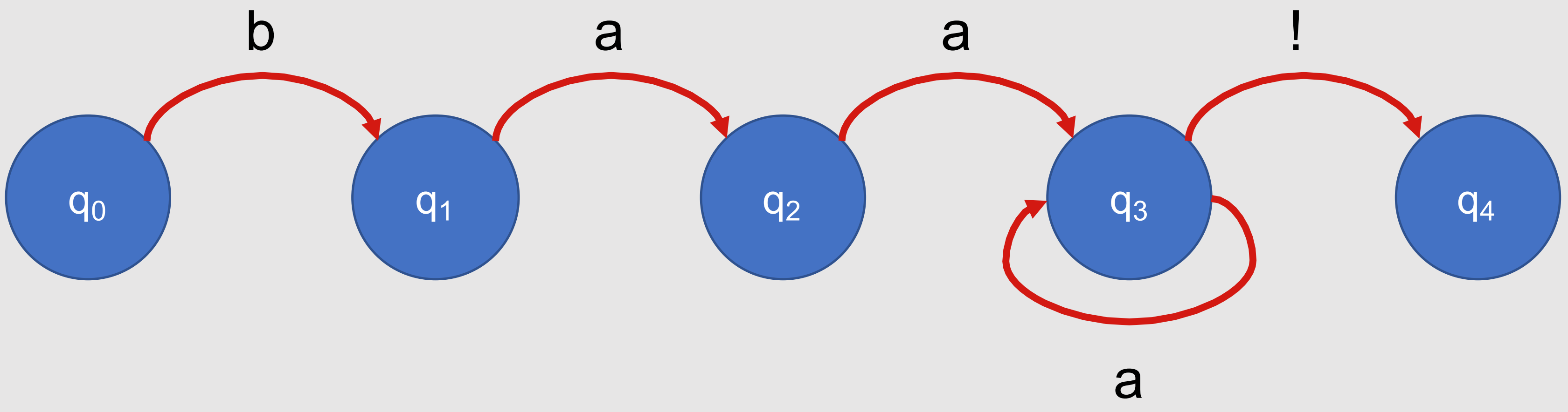


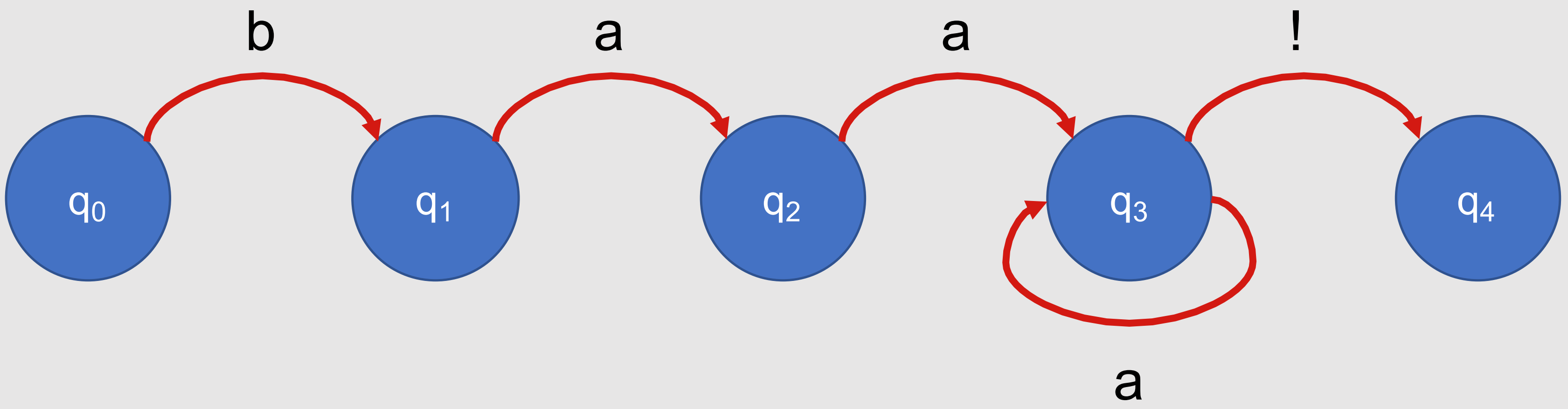
Next Item in Sequence

Currently in State

	b	a	!	<end>
q_0	q_1	☹	☹	☹
q_1	☹	q_2	☹	☹
q_2	☹	q_3	☹	☹
q_3	☹	q_3		
q_4				

Go to State

State transitions in FSAs can be represented using tables.



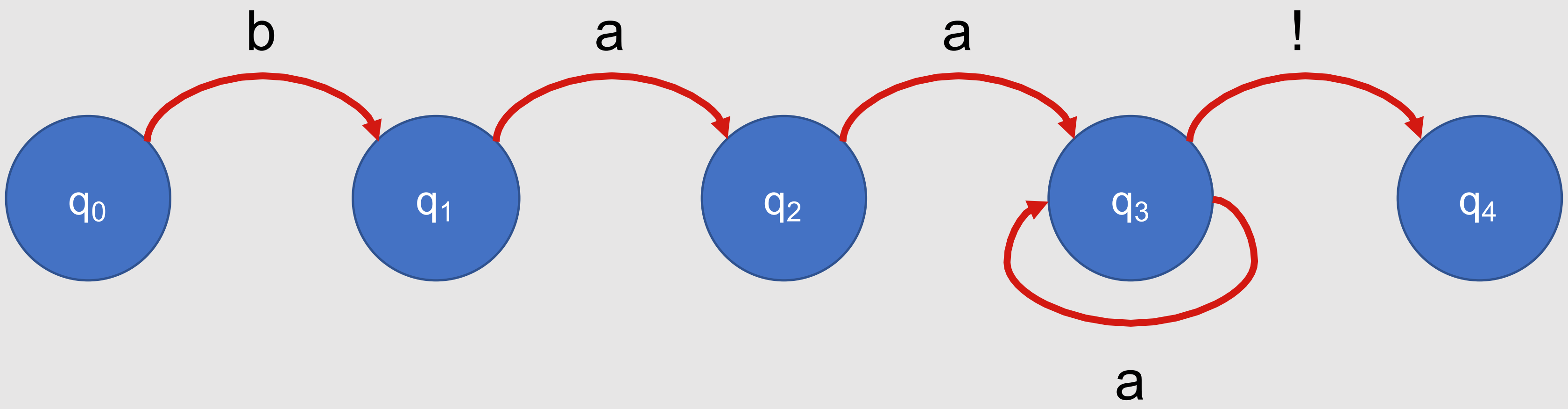
Next Item in Sequence

Currently in State

	b	a	!	<end>
q_0	q_1	☹	☹	☹
q_1	☹	q_2	☹	☹
q_2	☹	q_3	☹	☹
q_3	☹	q_3	q_4	
q_4				

Go to State

State transitions in FSAs can be represented using tables.



Next Item in Sequence

Currently in State

	b	a	!	<end>
q_0	q_1	☹	☹	☹
q_1	☹	q_2	☹	☹
q_2	☹	q_3	☹	☹
q_3	☹	q_3	q_4	☹
q_4	☹	☹	☹	☺

Accept!

State transition tables simplify the process of determining whether your input will be accepted by the FSA.

- For a given sequence of items to match, **begin in the start state** with the first item in the sequence
- **Consult the table** …is a transition to any other state permissible with the current item?
- If so, **move to the state indicated by the table**
- If you make it to the end of your sequence and to a final state, **accept**

Formal Algorithm

```
index ← beginning of sequence
current_state ← initial state of FSA
loop:
       if end of sequence has been reached:
              if current_state is an accept state:
                     return accept
              else:
                     return reject
       else if transition_table[current_state, sequence[index]] is empty:
              return reject
       else:
              current_state ← transition_table[current_state, sequence[index]]
              index ← index + 1
end
```

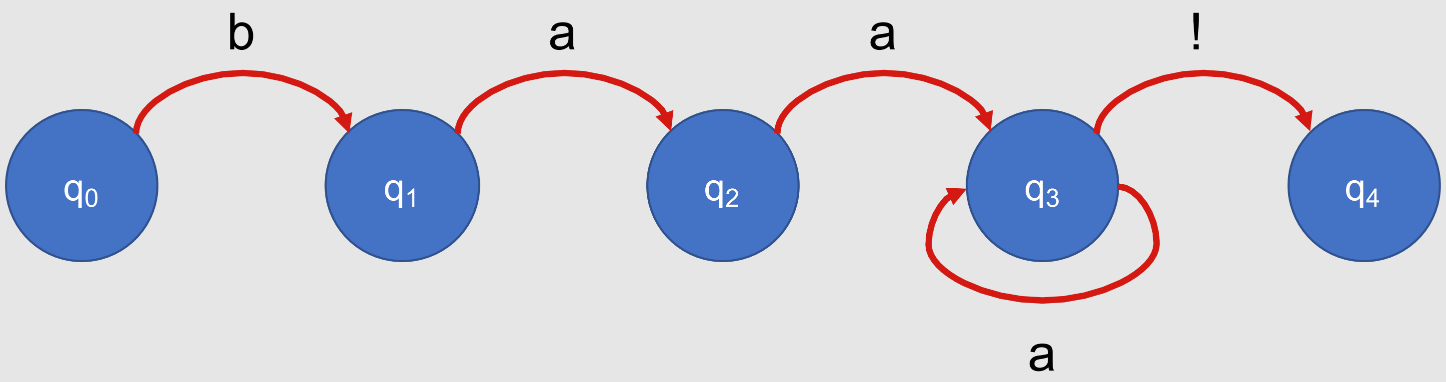
Deterministic vs. Non-Deterministic FSAs

Deterministic FSA: At each point in processing a sequence, there is one unique thing to do (no choices!)

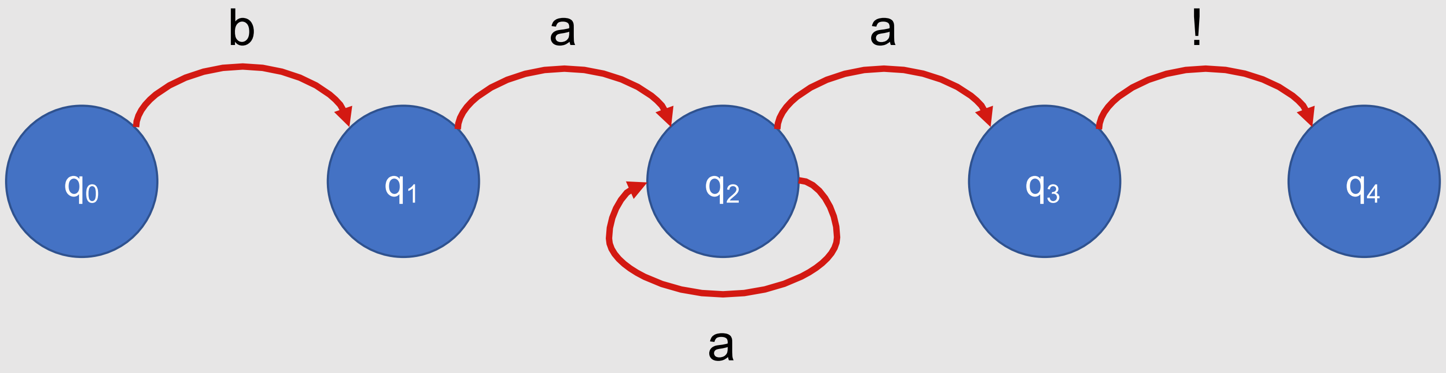
Non-Deterministic FSA: At one or more points in processing a sequence, there are multiple permissible next steps (choices!)

Deterministic or Non-Deterministic?

?

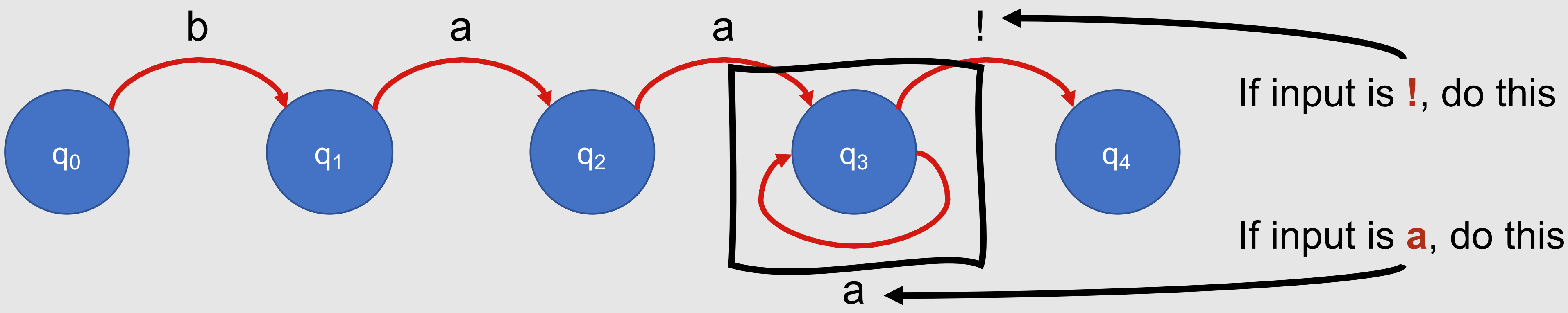


?

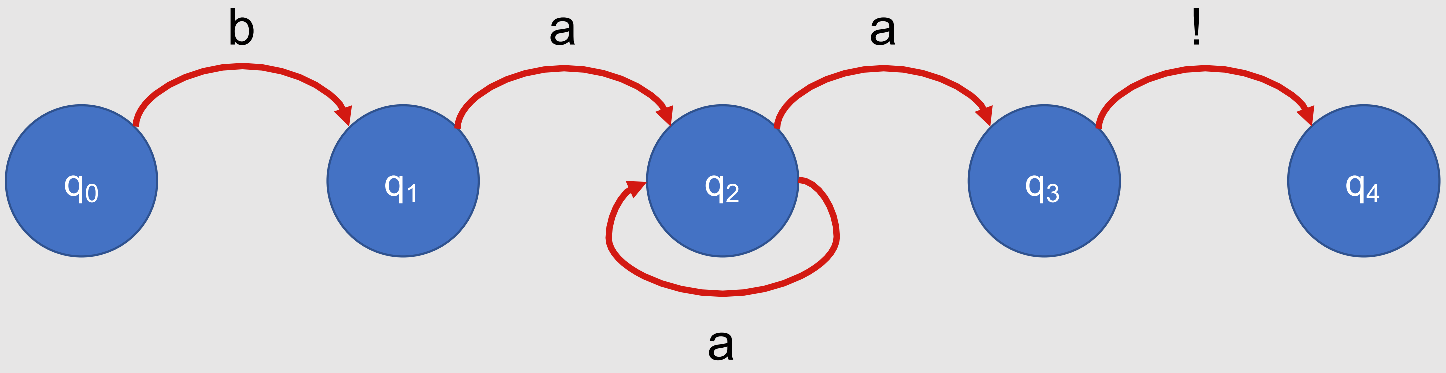


Deterministic or Non-Deterministic?

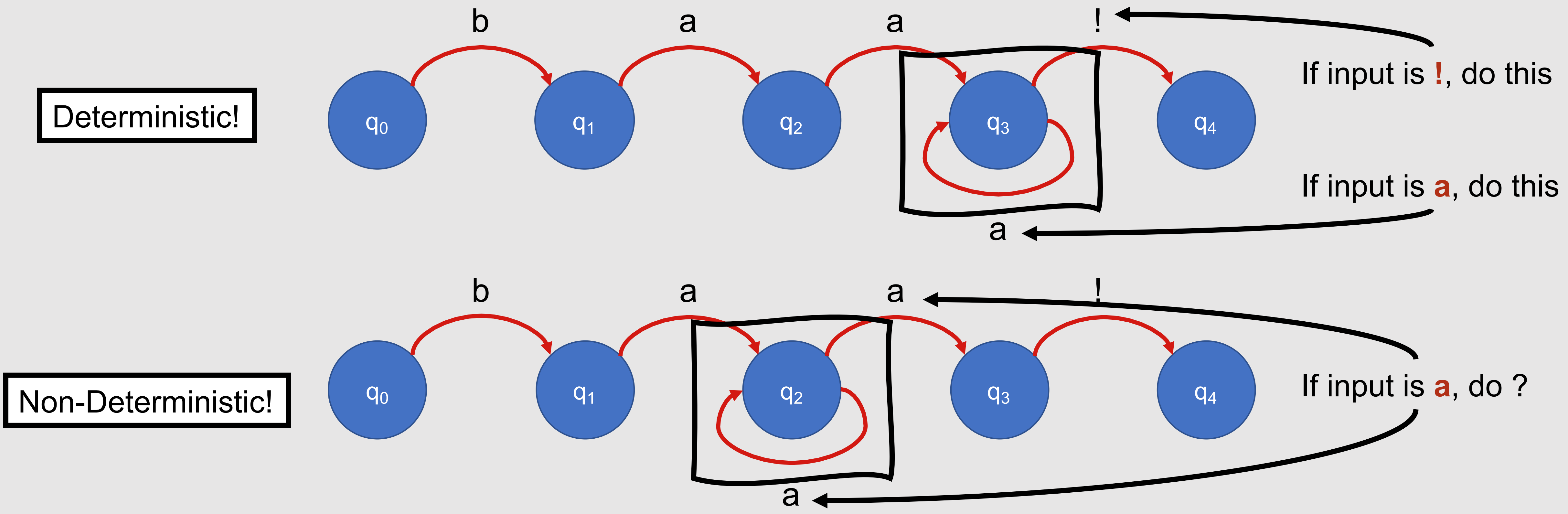
Deterministic!



?



Deterministic or Non-Deterministic?



Every non-deterministic FSA can be converted to a deterministic FSA.

- This means that both are equally powerful!
- Deterministic FSAs can accept as many languages as non-deterministic ones

Non-Deterministic FSAs: How to check for input acceptance?

- Two approaches:
  1. Convert the non-deterministic FSA to a deterministic FSA and then check that version
  2. Manage the process as a state-space search

Non-Deterministic FSA Search Assumptions

There exists at least one path through the FSA for an item that is part of the language defined by the machine

Not all paths directed through the FSA for an accept item lead to an accept state

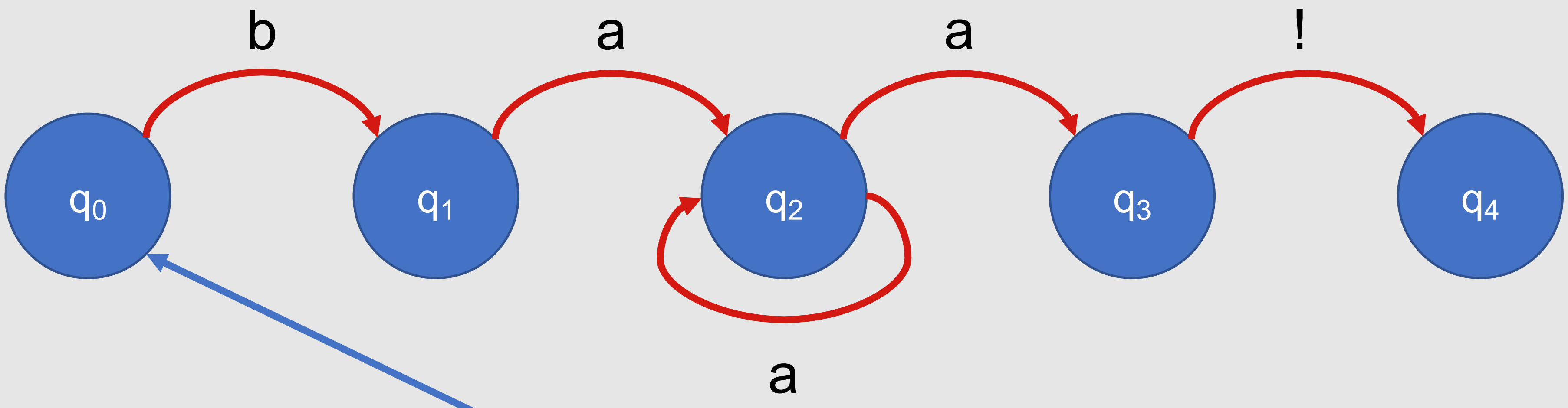
No paths through the FSA lead to an accept state for an item not in the language

Non-Deterministic FSA Search Assumptions

SUCCESS = PATH IS FOUND FOR A GIVEN ITEM THAT ENDS IN AN ACCEPT

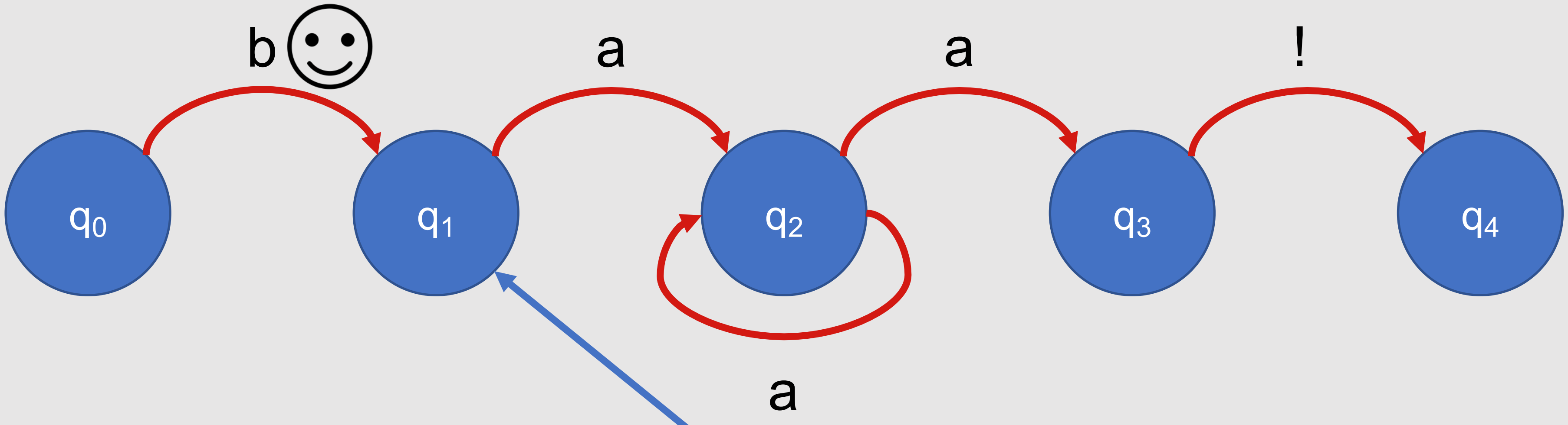
FAILURE = ALL POSSIBLE PATHS FOR A GIVEN ITEM LEAD TO FAILURE

Example: Non-Deterministic FSA Search



Test Input: baaa!

Example: Non-Deterministic FSA Search



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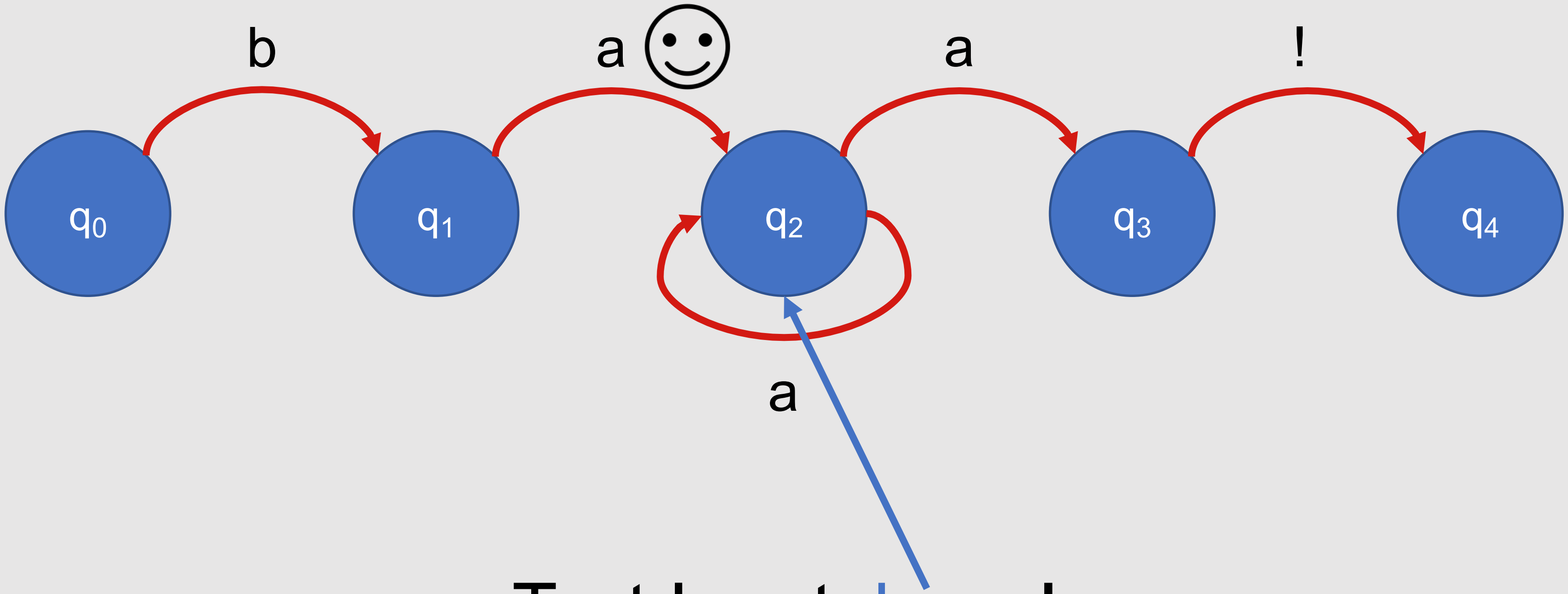
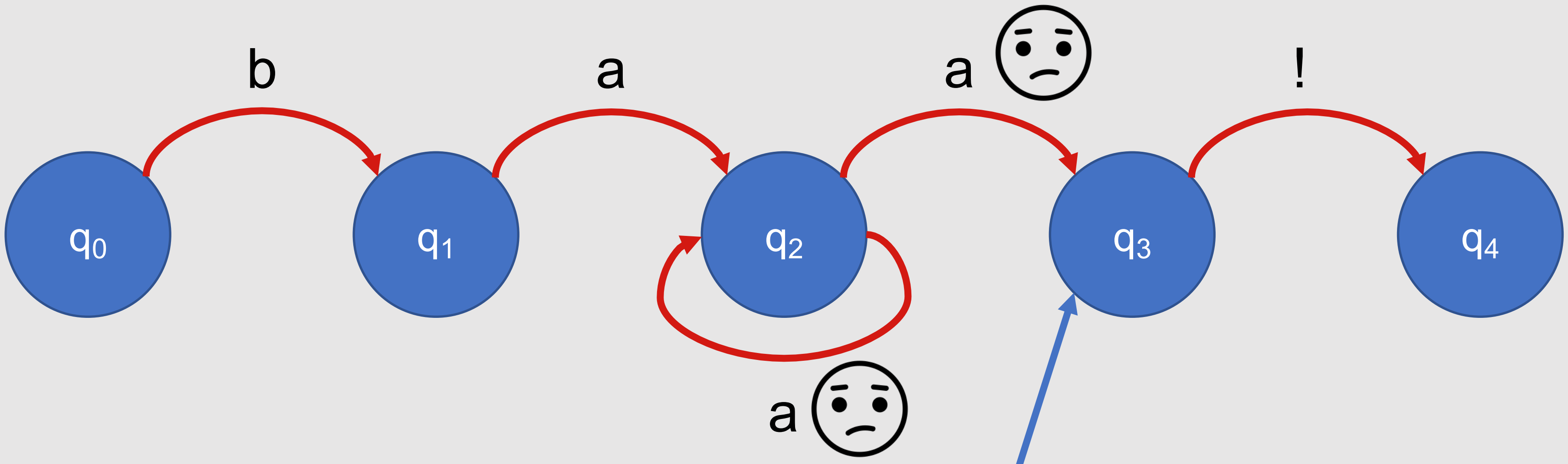
Example: Non-Deterministic FSA Search



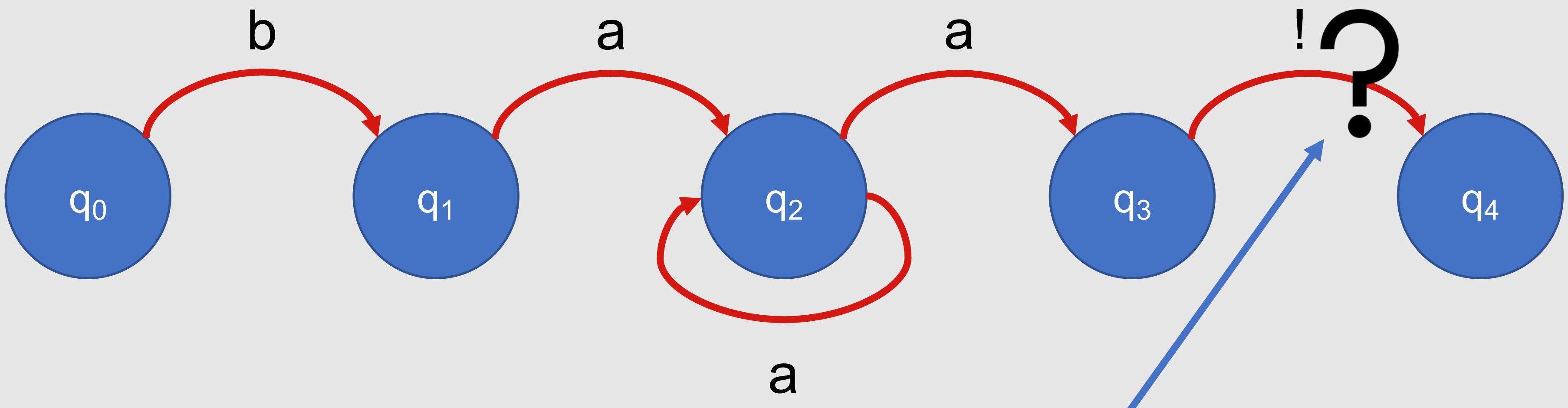
Test Input: baaa!

Example: Non-Deterministic FSA Search



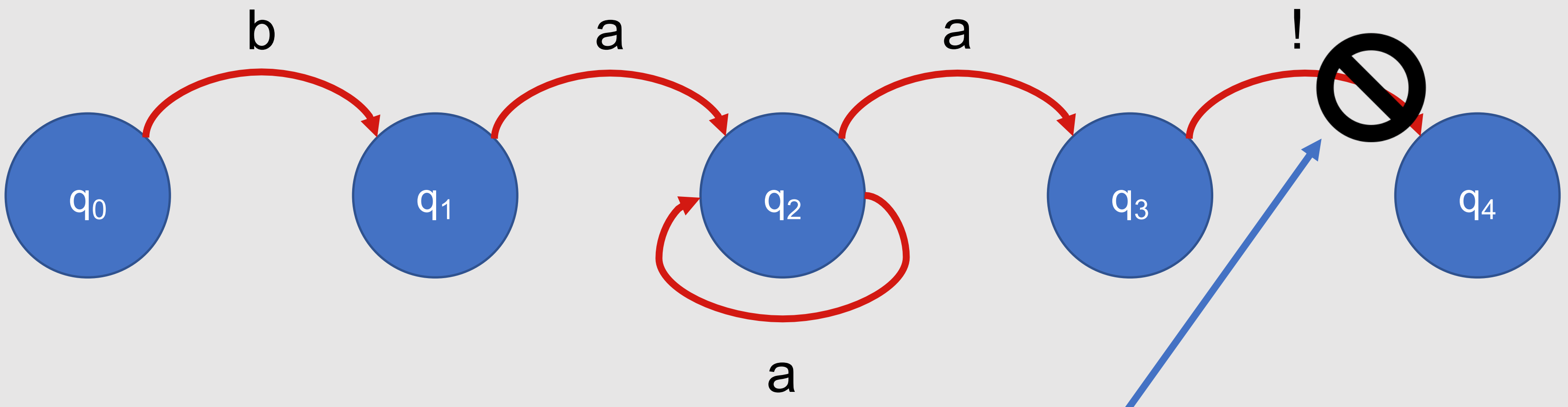
Test Input: baaa!

Example: Non-Deterministic FSA Search



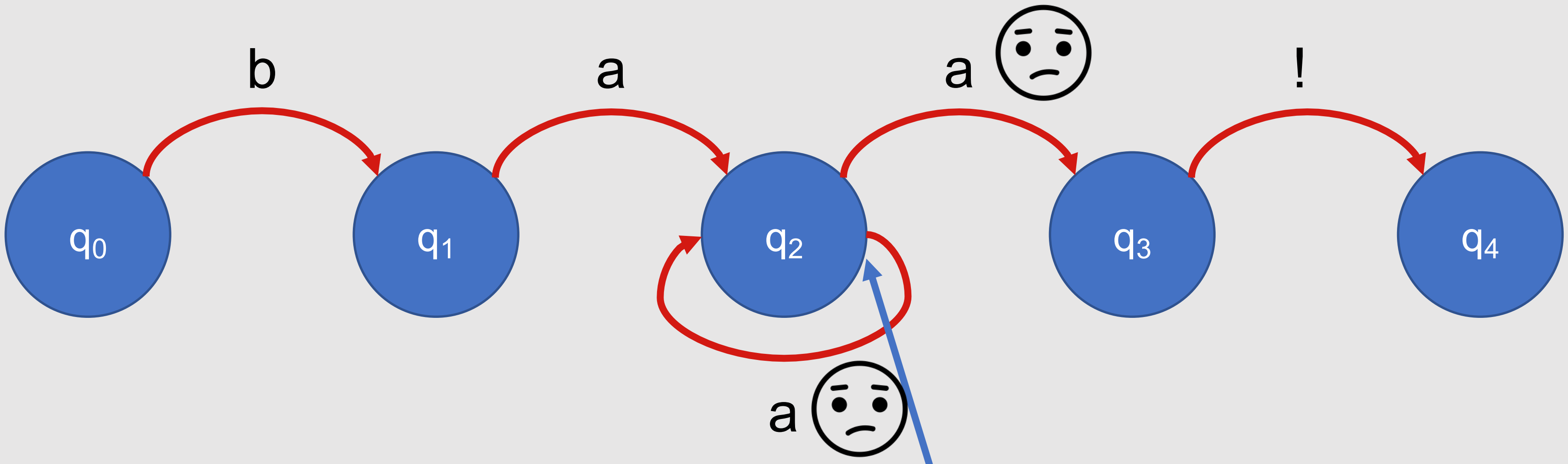
Test Input: baa**a**!

Example: Non-Deterministic FSA Search



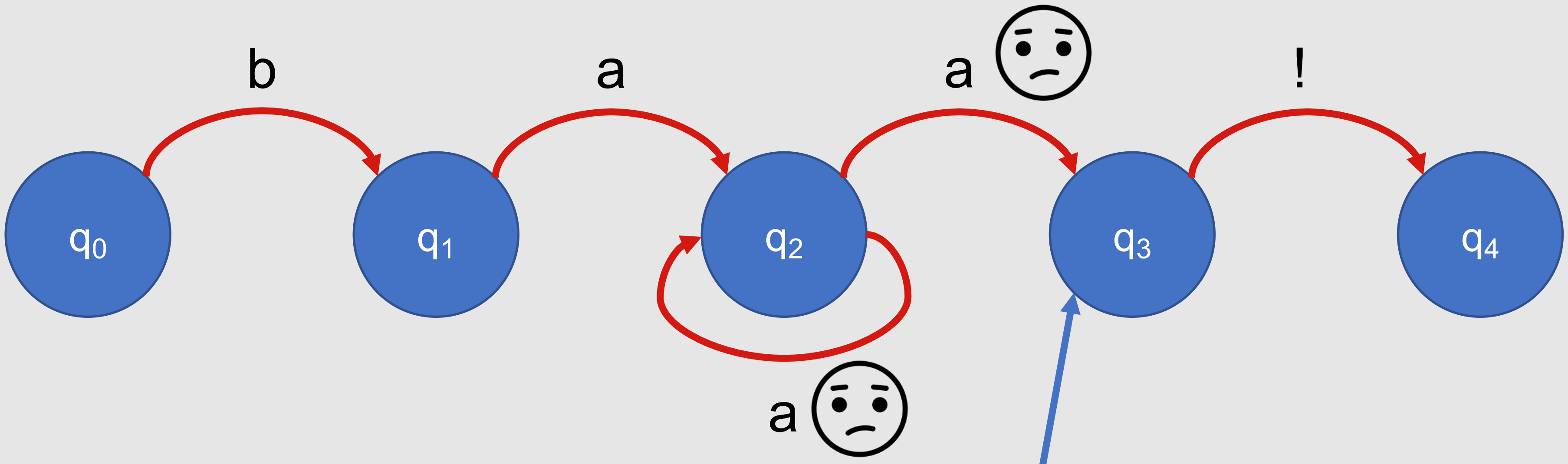
Test Input: baa**a**! ☹

Example: Non-Deterministic FSA Search



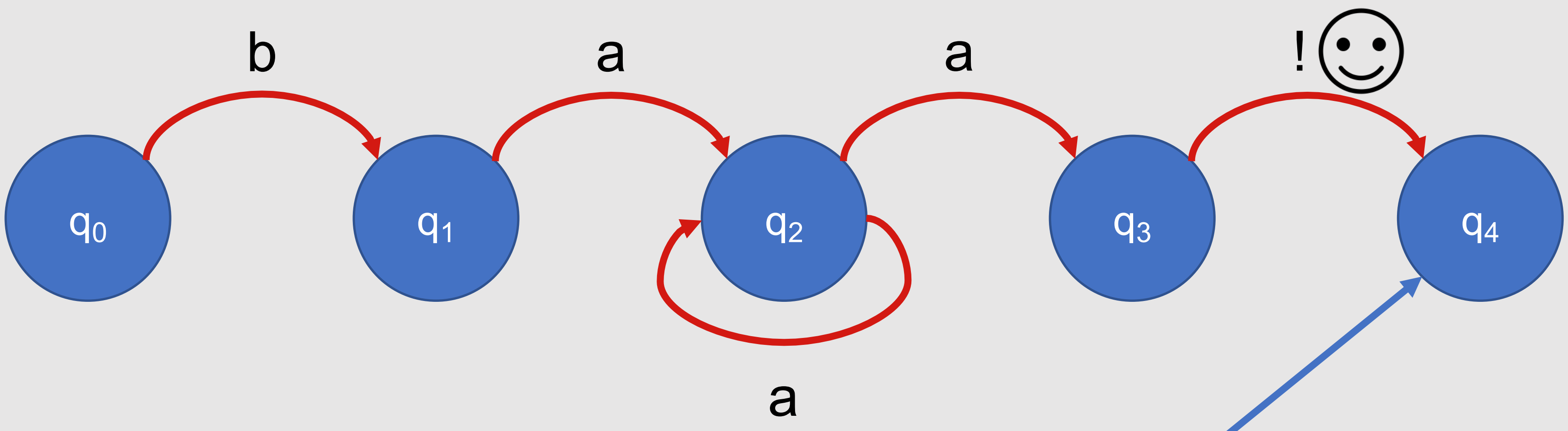
Test Input: baaa!

Example: Non-Deterministic FSA Search



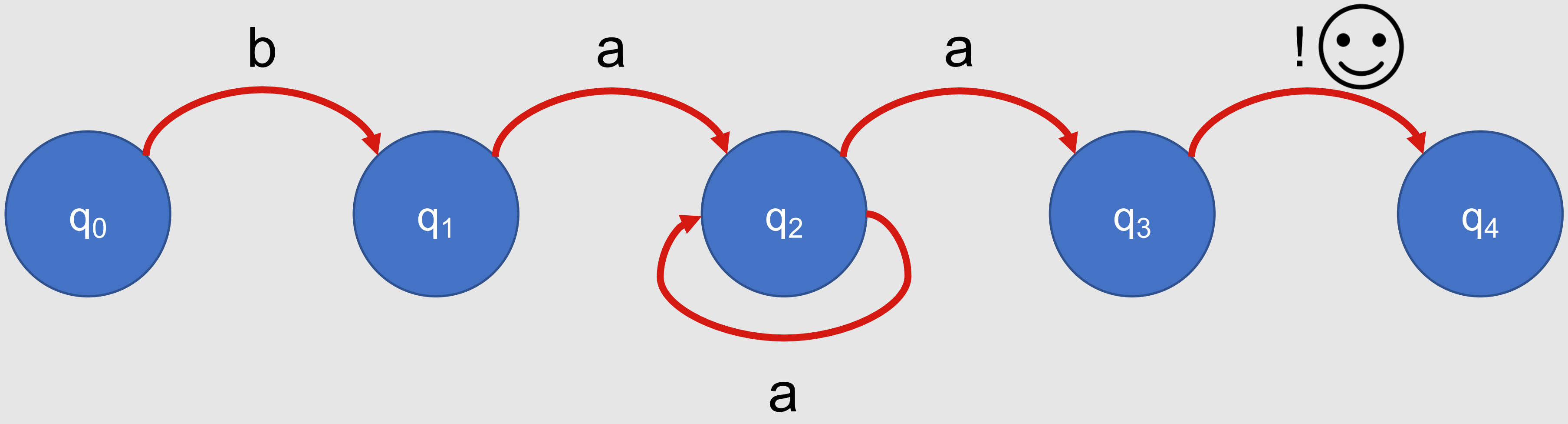
Test Input: baa**a**!

Example: Non-Deterministic FSA Search



Test Input: baaa!

Example: Non-Deterministic FSA Search



Test Input: baaa!

Non-Deterministic FSA Search

- States in the search space are pairings of sequence indices and states in the FSA
- By keeping track of which states have and have not been explored, we can systematically explore all the paths through an FSA given an input

Compositional FSAs

- You can apply set operations to any FSA
  - Union
  - Concatenation
  - Negation
    - For non-deterministic FSAs, first convert to a deterministic FSA
  - Intersection
- To do so, you may need to utilize an ϵ transition
  - ϵ transition: Move from one state to another without consuming an item from the input sequence

Summary: Finite State Automata

- FSAs are computational models that describe regular languages
- To determine whether an input item is a member of an FSA's language, you can process it sequentially from the start to (hopefully) the final state
- State transitions in FSAs can be represented using tables
- FSAs can be either deterministic or non-deterministic

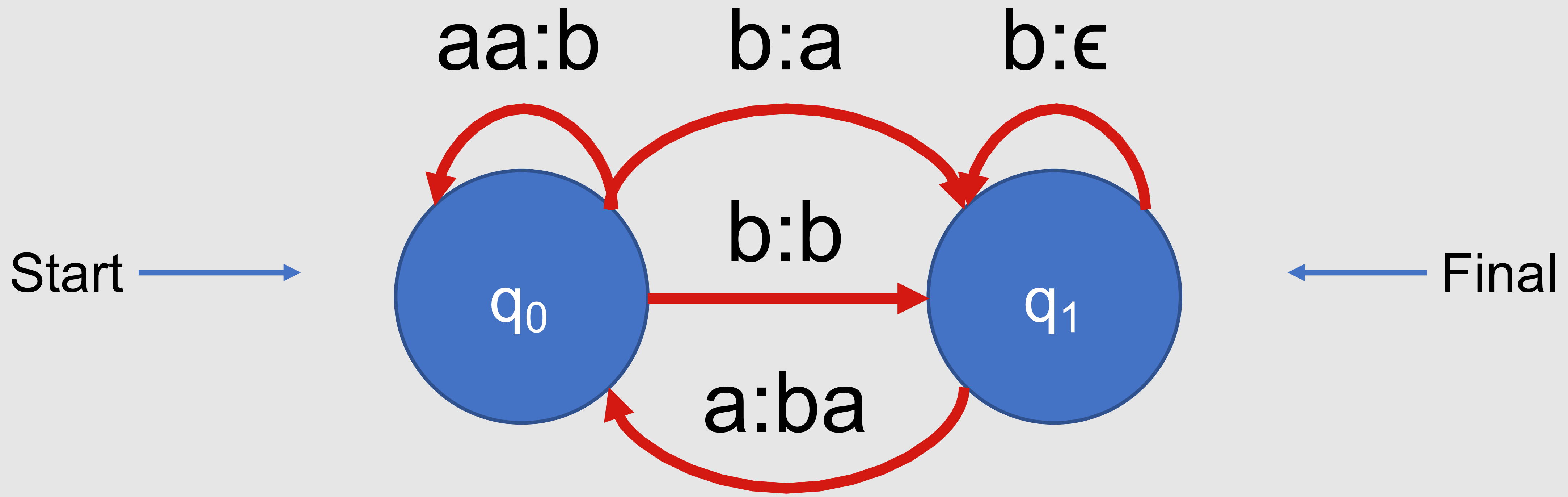
What are finite state transducers?

Finite State Transducer (FST): A type of FSA that describes mappings between two sets of items

This means that FSTs recognize or generate pairs of items

FSAs can be converted to FSTs by labeling each arc with two items (e.g., **a**:**b** for an input of **a** and and an output of **b**)

Example: Simple FST



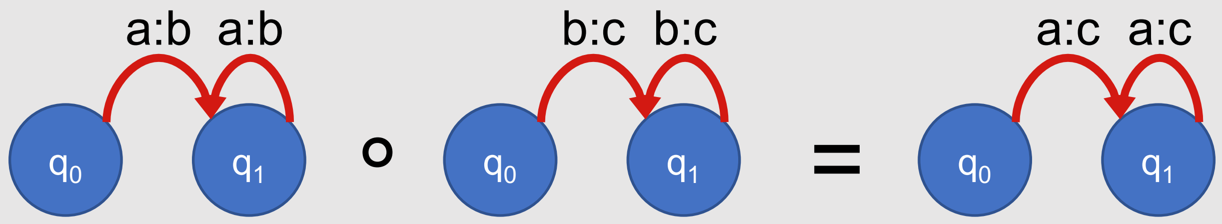
Formal Definition

- A finite state transducer can be specified by enumerating the following properties:
  - The set of states, ***Q***
  - A finite input alphabet, $\boldsymbol{\Sigma}$
  - A finite output alphabet, $\boldsymbol{\Delta}$
  - A start state, q_0
  - A set of accept/final states, $F \subseteq Q$
  - A transition function or transition matrix between states, $\delta(q,i)$
  - An output function giving the set of possible outputs for each state and input, $\sigma(q,i)$
- $\delta(q,i)$: Given a state $q \in Q$ and input $i \in \Sigma$, **$\delta(q,i)$ returns a new state $q' \in Q$**.

Formal Properties

Composition: Letting T_1 be an FST from I_1 to O_1 and letting T_2 be an FST from I_2 to O_2, the two FSTs can be composed such that the resulting FST maps directly from I_1 to O_2.

Inversion: Letting T be an FST that maps from I to O, its inversion (T^{-1}) will map from O to I.



Deterministic vs. Non-Deterministic FSTs

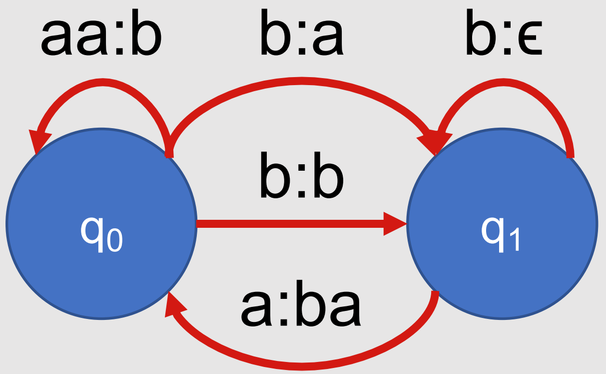
Just like FSAs, **FSTs can be non-deterministic** …one input can be translated to many possible outputs!

Unlike FSAs, **not all non-deterministic FSTs can be converted to deterministic FSTs**

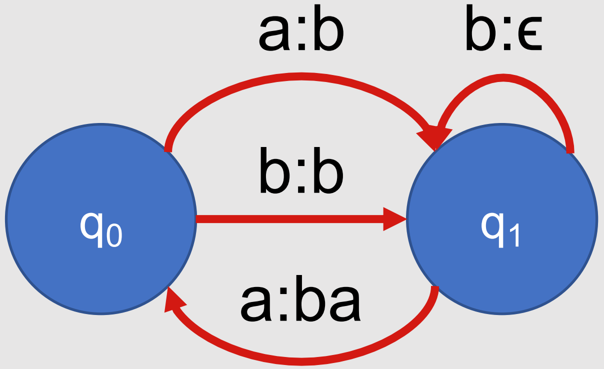
FSTs with underlying deterministic FSAs (at any state, a given input maps to at most one transition out of the state) are called **sequential transducers**

Examples: Non-Deterministic and Sequential Transducers

Non-Deterministic



Sequential



Remember morphology?

- **Morphemes**:
  - Small meaningful units that make up words
  - **Stems**: The core meaning-bearing units
  - **Affixes**: Bits and pieces that adhere to stems and add additional information
    - -ed
    - -ing
    - -s
- Morphological parsing is a classic use case for FSTs

Morphological Parsing

- The task of recognizing the component morphemes of words (e.g., foxes → fox + es) and building structured representations of those components

Why is morphological parsing necessary?

Morphemes can be **productive**

- Example: -ing attaches to almost every verb, including brand new words
  - “Why are you Instagramming that?”

Some languages are very **morphologically complex**

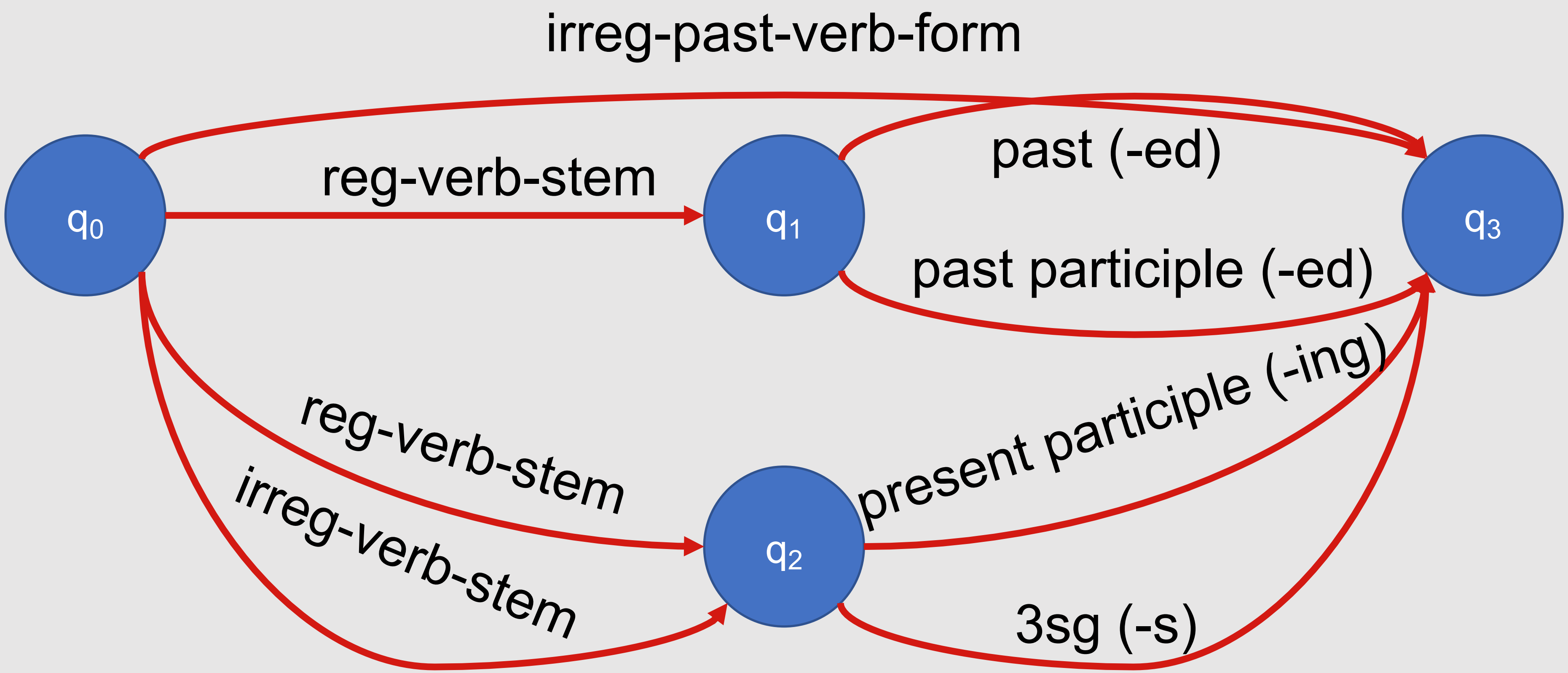
- Uygarlastiramadiklarimizdanmissinizcasina
  - Uygar ‘civilized’ + las ‘become’
  - + tir ‘cause’ + ama ‘not able’
  - + dik ‘past’ + lar ‘plural’
  - + imiz ‘p1pl’ + dan ‘abl’
  - + mis ‘past’ + siniz ‘2pl’ + casina ‘as if’

Finite State Morphological Parsing

Goal: Take input surface realizations and produce morphological parses as output

Surface Text	Morphological Parse
cats	cat +N +PL
cat	cat +N +SG
cities	city +N +PL
geese	goose +N +PL
goose	goose +N +SG
merging	merge +V +PresPart
caught	catch +V +Past

Example Morphological Lexicon

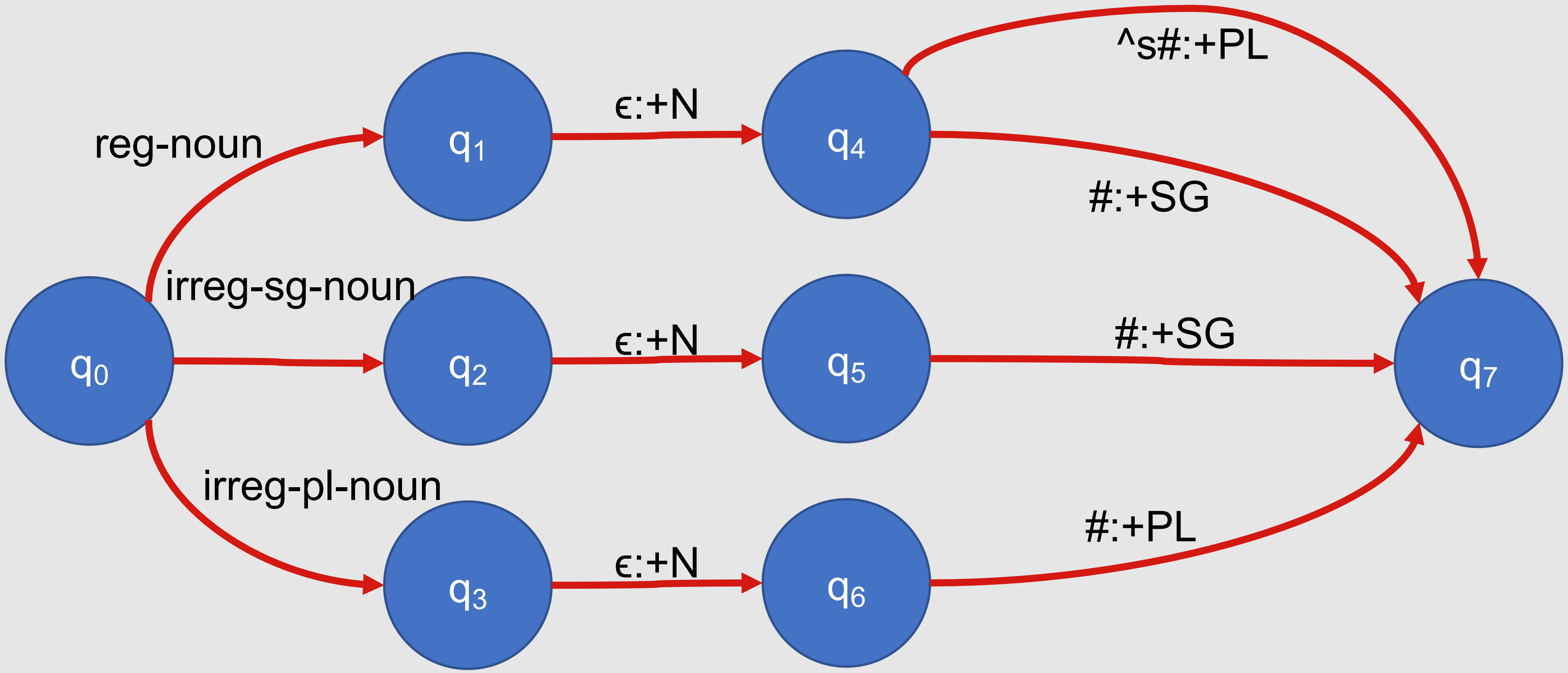


Finite State Morphological Parsing

- Two sets of items:
  - Surface form (input text)
  - Lexical form (morphological parse)

Finite State Morphological Parsing

reg-noun	irreg-pl-noun	irreg-sg-noun
fox	g o:e o:e s e	goose
cat		



Finite State Morphological Parsing

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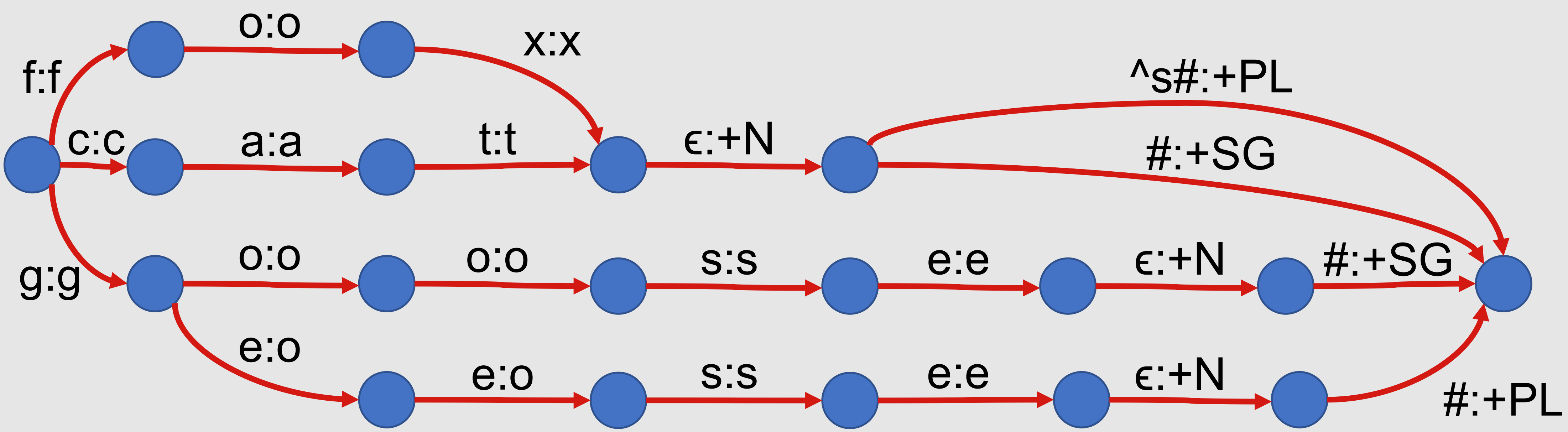


Summary: Finite State Transducers

- FSTs are FSAs that describe mappings between two sets
- Although all non-deterministic FSAs can be converted to deterministic versions, all non-deterministic FSTs cannot
- FSTs with underlying deterministic FSAs are called sequential transducers
- FSTs are particularly useful for morphological parsing

What are Hidden Markov Models (HMMs)?

Probabilistic generative models for **sequences**

Make predictions based on an underlying set of **hidden states**

How does sequence labeling differ from other types of classification?

- A lot of machine learning addresses the problem of classifying instances into a predefined number of classes
  - Decision Trees
  - Naïve Bayes
  - Logistic Regression
  - (Some) Neural Networks
  - Support Vector Machines

Spam

Not Spam

Dear Esteemed Professor Dr. ***Natalie Parde***,
I am interested in applying to ***University of Illinois – Chicago*** for a **Ph.D.** in **Computer Science** in the area of ***Artificial Intelligence*** and ***Natural Language Processing***. I read your recent paper ***"Enriching Neural Models with Targeted Features for Dementia Detection***" and see that you are interested in **Neural Models** and **Dementia Detection**.…

Standard Classification Assumption: Individual cases are disconnected and independent.

However, many NLP problems do not satisfy this assumption.

Instead, they involve many interconnected decisions, each of which resolve different ambiguities despite being mutually dependent.

For these problems, different learning and inference techniques are needed!

Sequence Labeling

- Many NLP problems can be viewed as **sequence labeling** tasks.
- Objective: Find the label for the next item, based on the labels of other items in the sequence.

verb determiner

Give me a **break**!

pronoun noun

verb noun

Did the window **break**?

determiner verb

Applications that can benefit from sequence labeling?

- Named entity recognition
- Semantic role labeling
- Genome analysis

person organization

Natalie Parde works at the **University of Illinois at Chicago** and lives in **Chicago, Illinois**.

location

agent source destination

Natalie drove for 15 hours from **Dallas** to **Chicago** in her trusty hail-damaged **Honda Accord**.

instrument

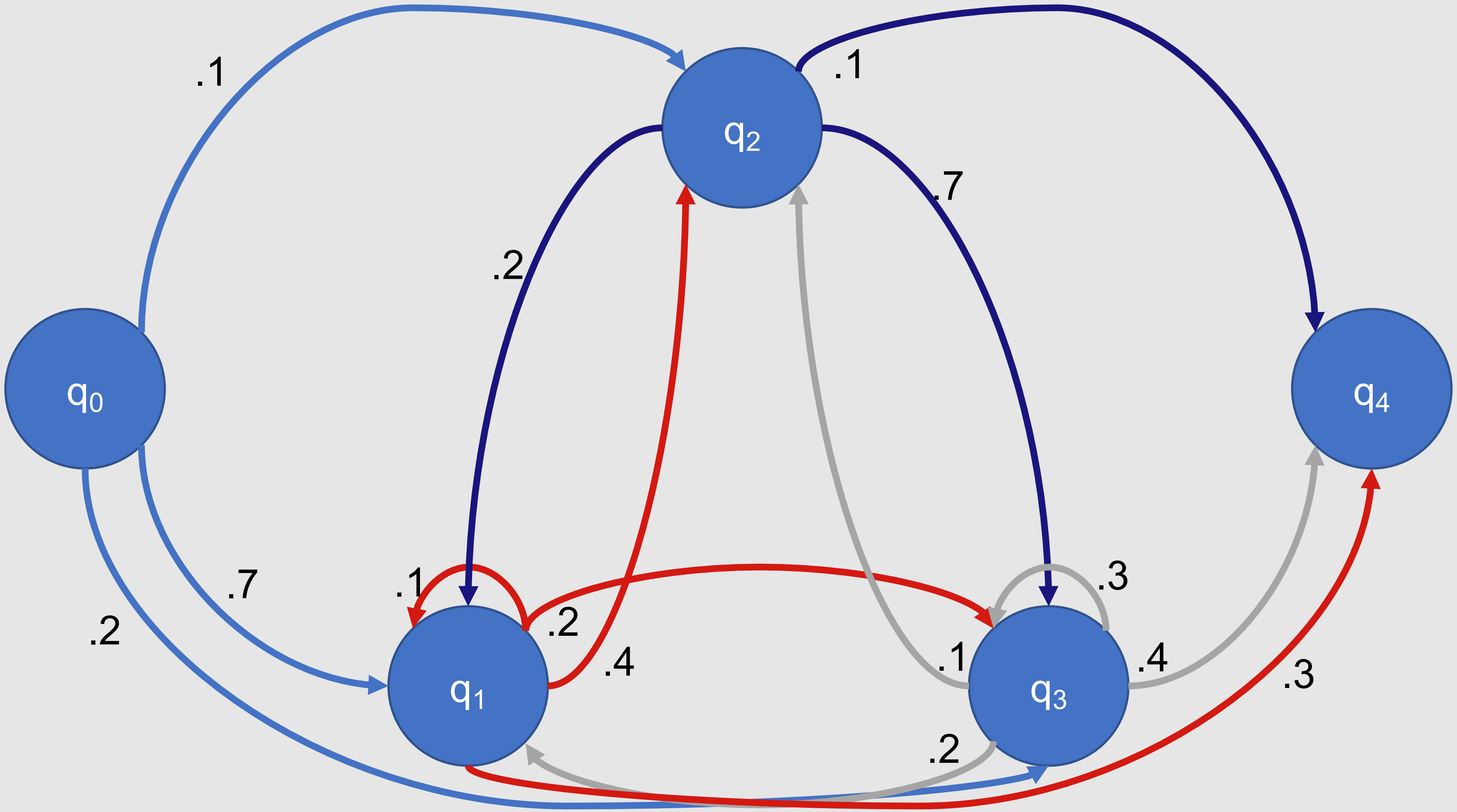
Probabilistic Sequence Models

- Allow uncertainties to be integrated over multiple, interdependent classifications
- These classifications collectively determine the most likely global assignment
- Two standard models:
  - Hidden Markov Models
  - Conditional Random Fields

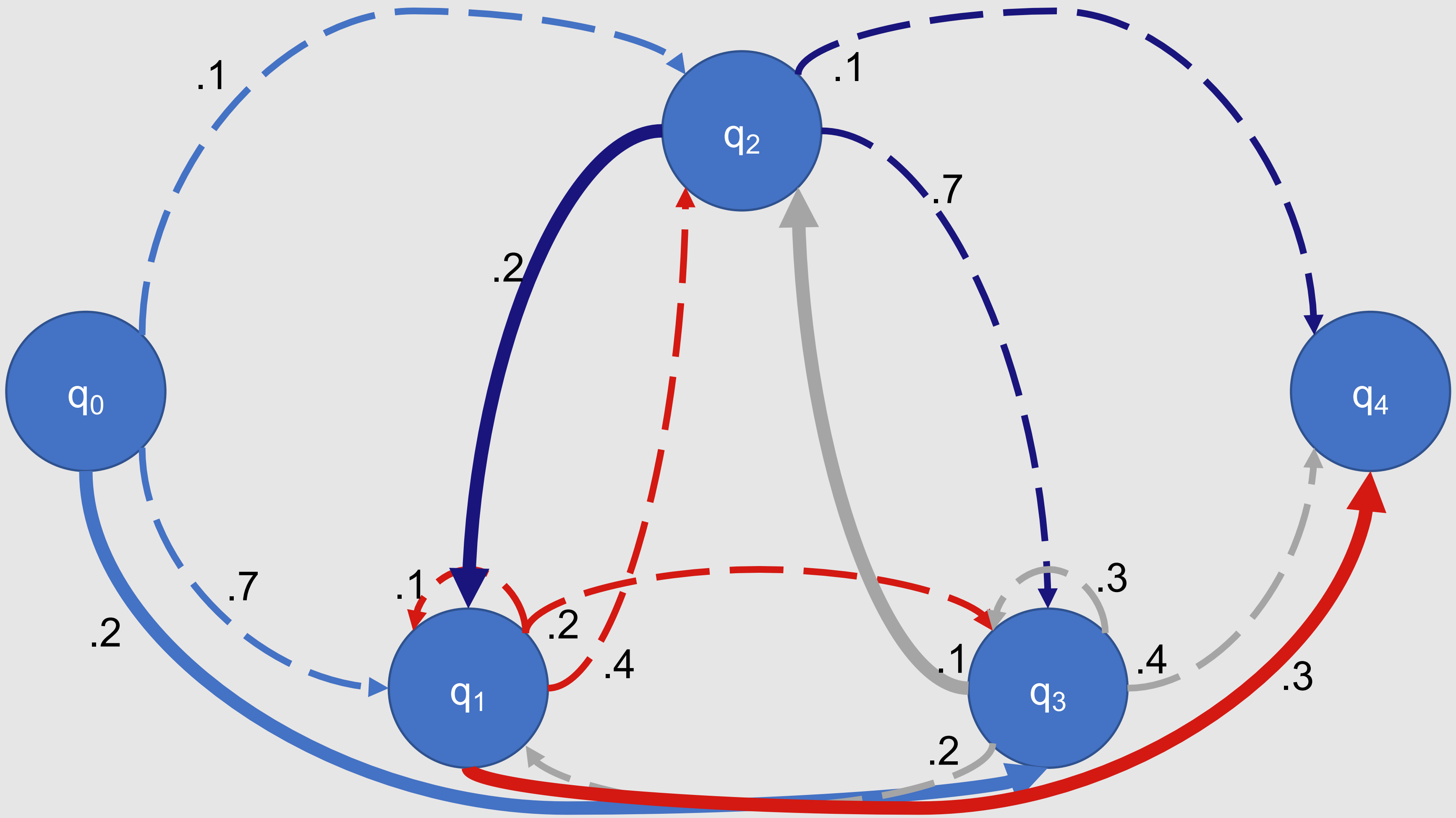
What are Markov Models?

- **Finite state automata with probabilistic state transitions**
- Markov Property: The future is independent of the past, given the present.
  - In other words, the next state only depends on the current state …it is independent of previous history.
- Also referred to as **Markov Chains**

Sample Markov Model



Sample Markov Model

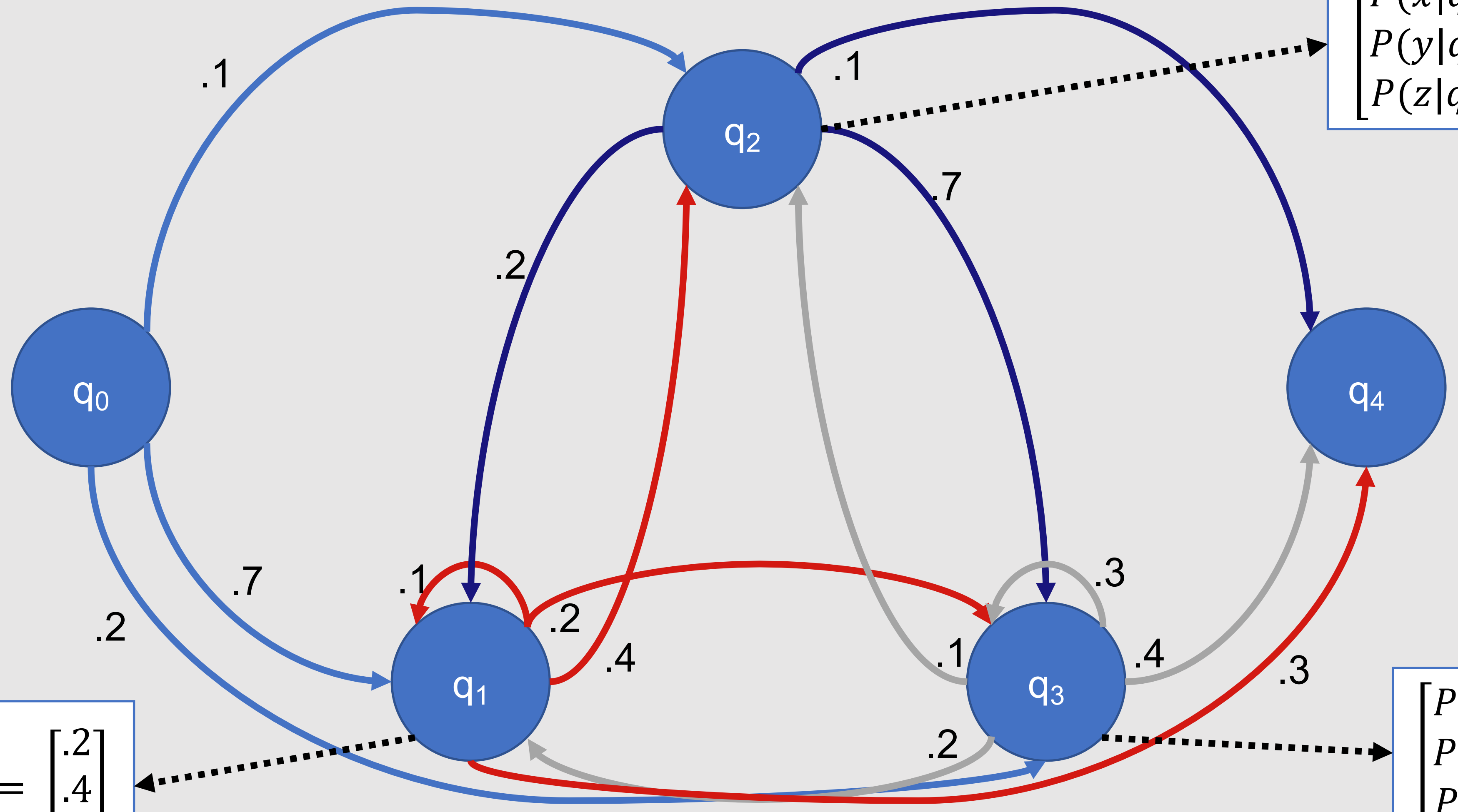


$P(q_3\ q_2\ q_1\ q_4)$
$= .2 * .1 * .2 * .3$
$= .0012$

Hidden Markov Models

- Probabilistic generative models for sequences
- Assume an underlying set of hidden (unobserved) states in which the model can be
- Assume probabilistic transitions between states over time
- Assume probabilistic generation of items (e.g., tokens) from states

Sample Hidden Markov Model



$$\begin{bmatrix} P(x|q_2) \\ P(y|q_2) \\ P(z|q_2) \end{bmatrix} = \begin{bmatrix} .1 \\ .4 \\ .5 \end{bmatrix}$$

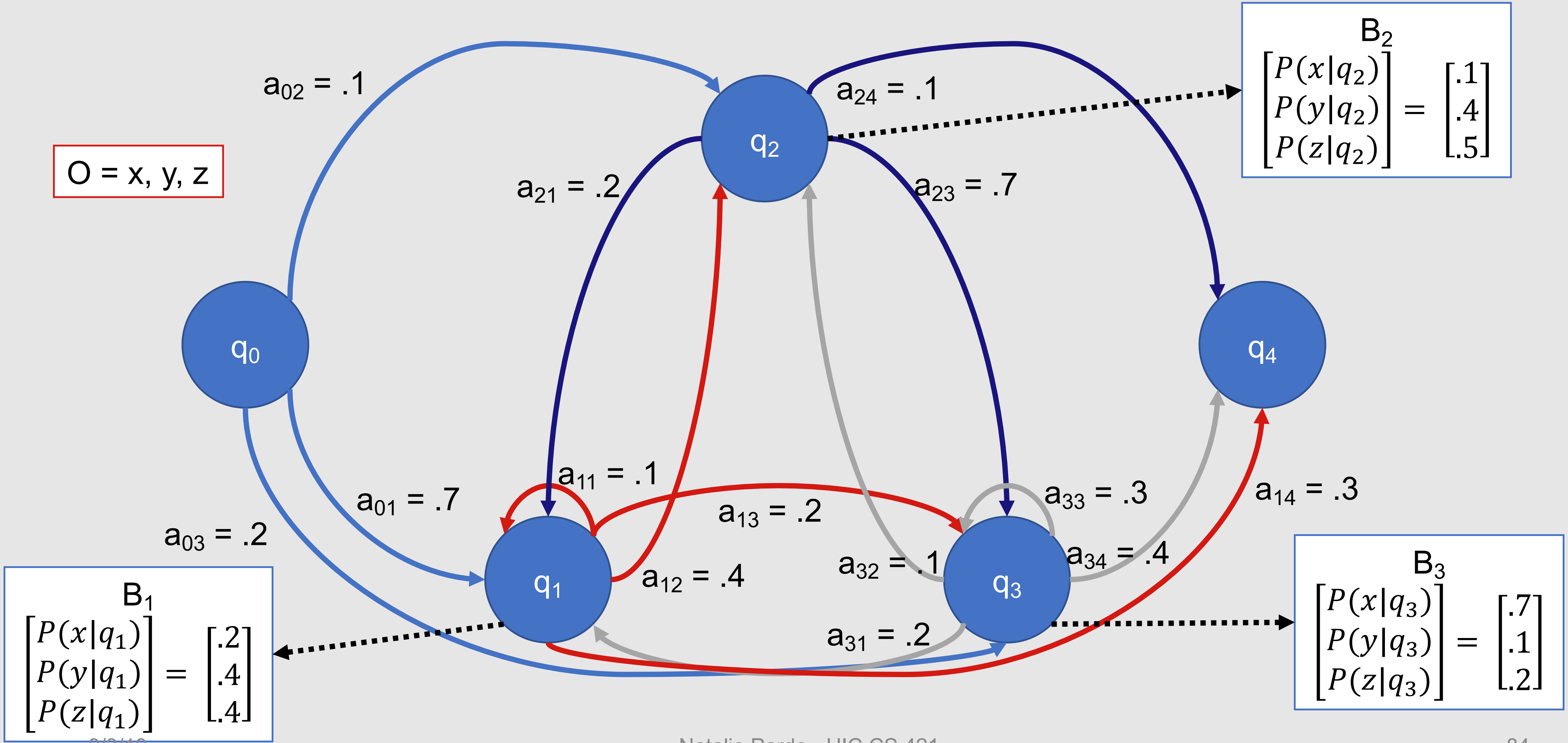
$$\begin{bmatrix} P(x|q_1) \\ P(y|q_1) \\ P(z|q_1) \end{bmatrix} = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix}$$

$$\begin{bmatrix} P(x|q_3) \\ P(y|q_3) \\ P(z|q_3) \end{bmatrix} = \begin{bmatrix} .7 \\ .1 \\ .2 \end{bmatrix}$$

Formal Definition

- A hidden Markov model can be specified by enumerating the following properties:
  - The set of states, ***Q***
  - A transition probability matrix, ***A***, where each a_{ij} represents the probability of moving from state *i* to state *j*, such that $\sum_{j=1}^{n} a_{ij} = 1 \; \forall i$
  - A sequence of T observations, ***O***, each drawn from a vocabulary V = $v_1, v_2, \ldots, v_V$
  - A sequence of observation likelihoods, ***B***, also called emission probabilities, each expressing the probability of an observation o_t being generated from a state *i*
  - A start state, q_0, and final state, q_F, that are not associated with observations, together with transition probabilities out of q_0 and into q_F

Sample Hidden Markov Model



Corresponding Transition Matrix

	q0	q1	q2	q3	q4
q0	N/A	.7	.1	.2	N/A
q1	N/A	.1	.4	.2	.3
q2	N/A	.2	N/A	.7	.1
q3	N/A	.2	.1	.3	.4
q4	N/A	N/A	N/A	N/A	N/A

Can we use HMMs to generate text?

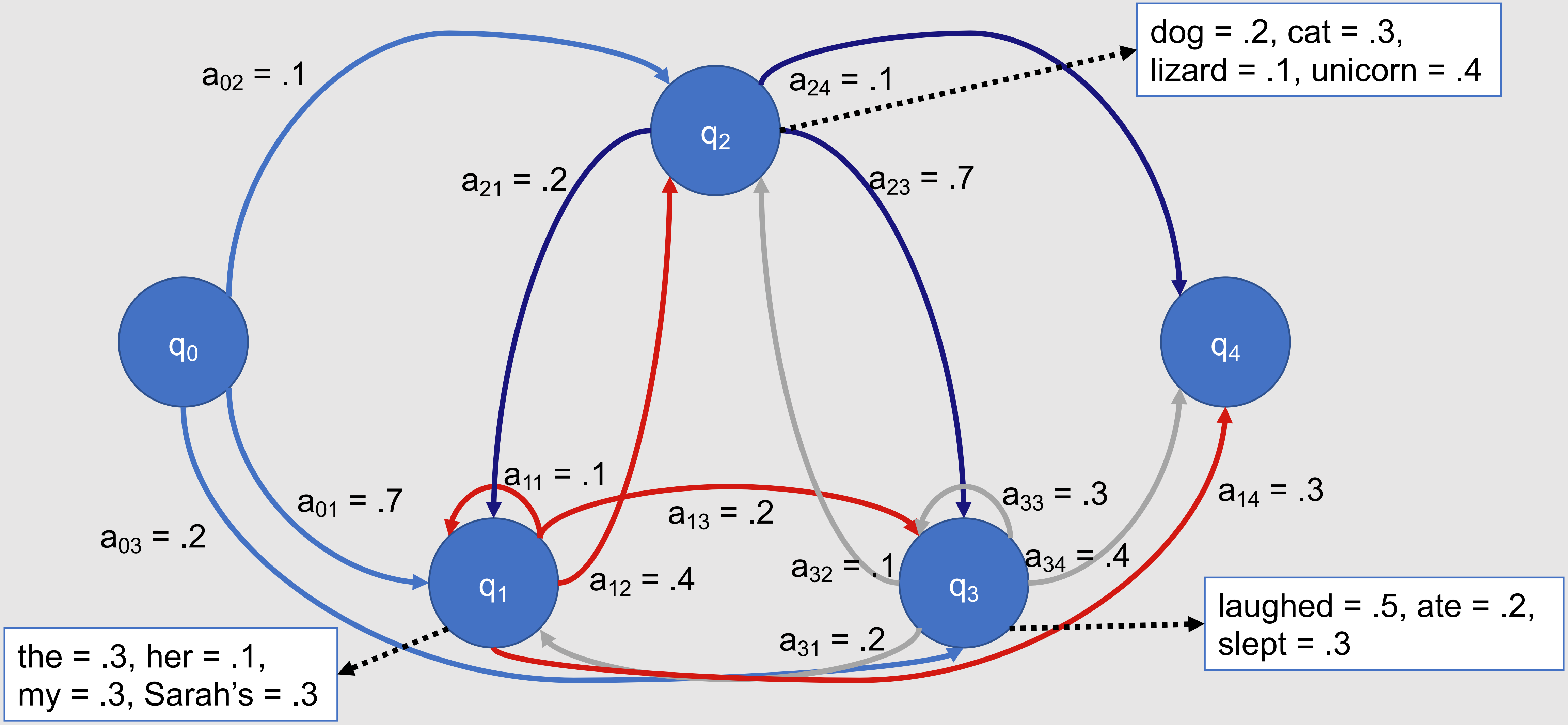
- Sure!
- More generally, you can generate a sequence of T observations: $O = o_1, o_2, \ldots, o_T$

Begin in the start state

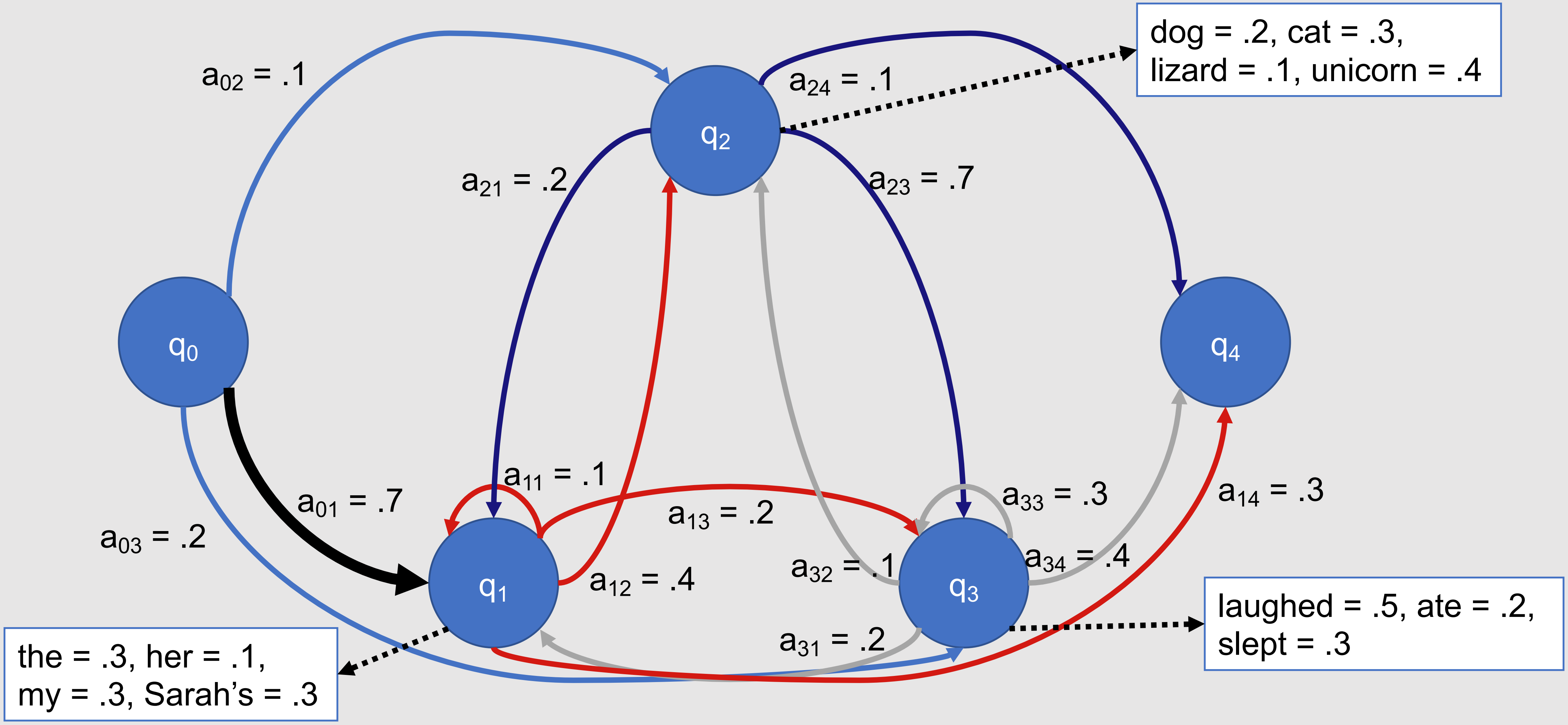
For t in [0, …, T]:

> *Randomly select a new state based on the transition distribution for the current state*
>
> *Randomly select an observation from the new state based on the observation distribution for that state*

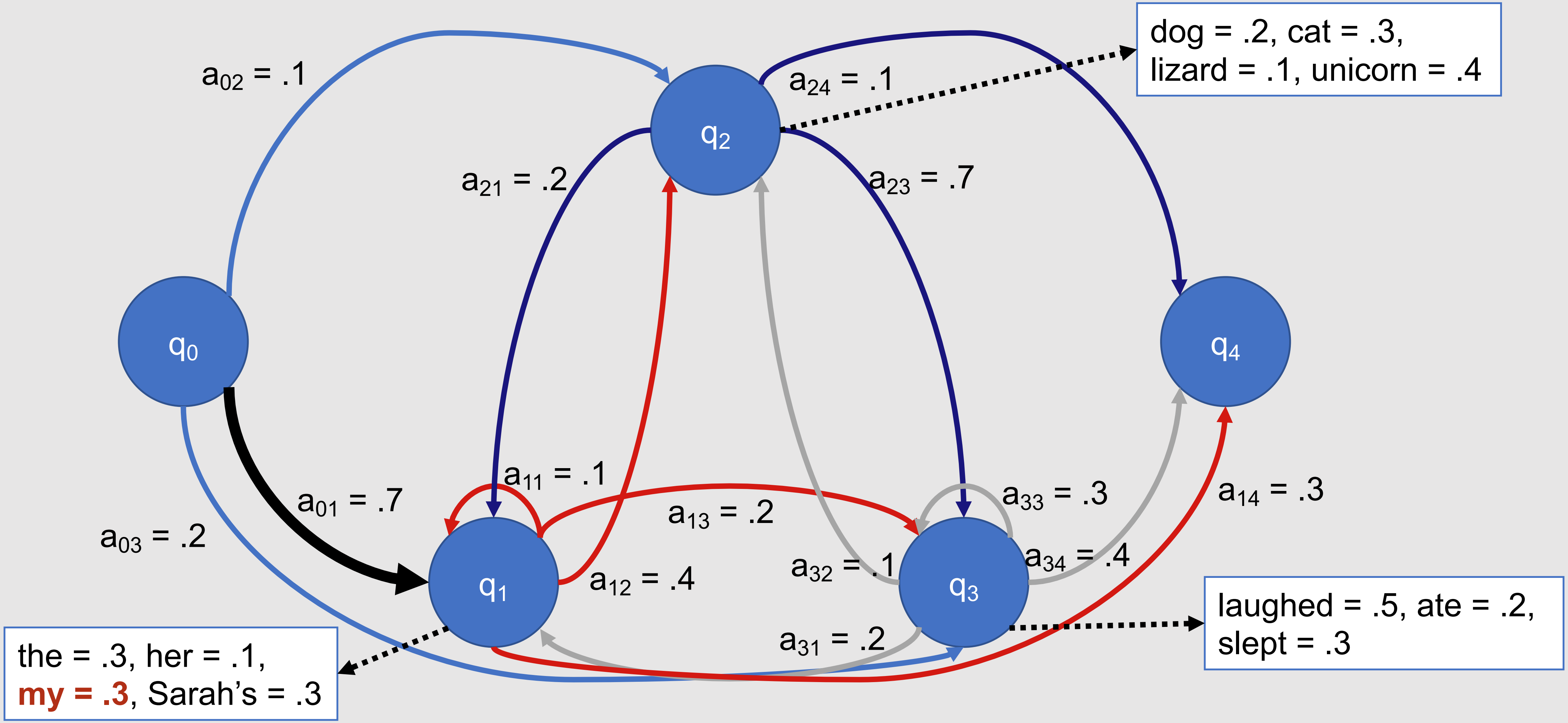
Sample Text Generation



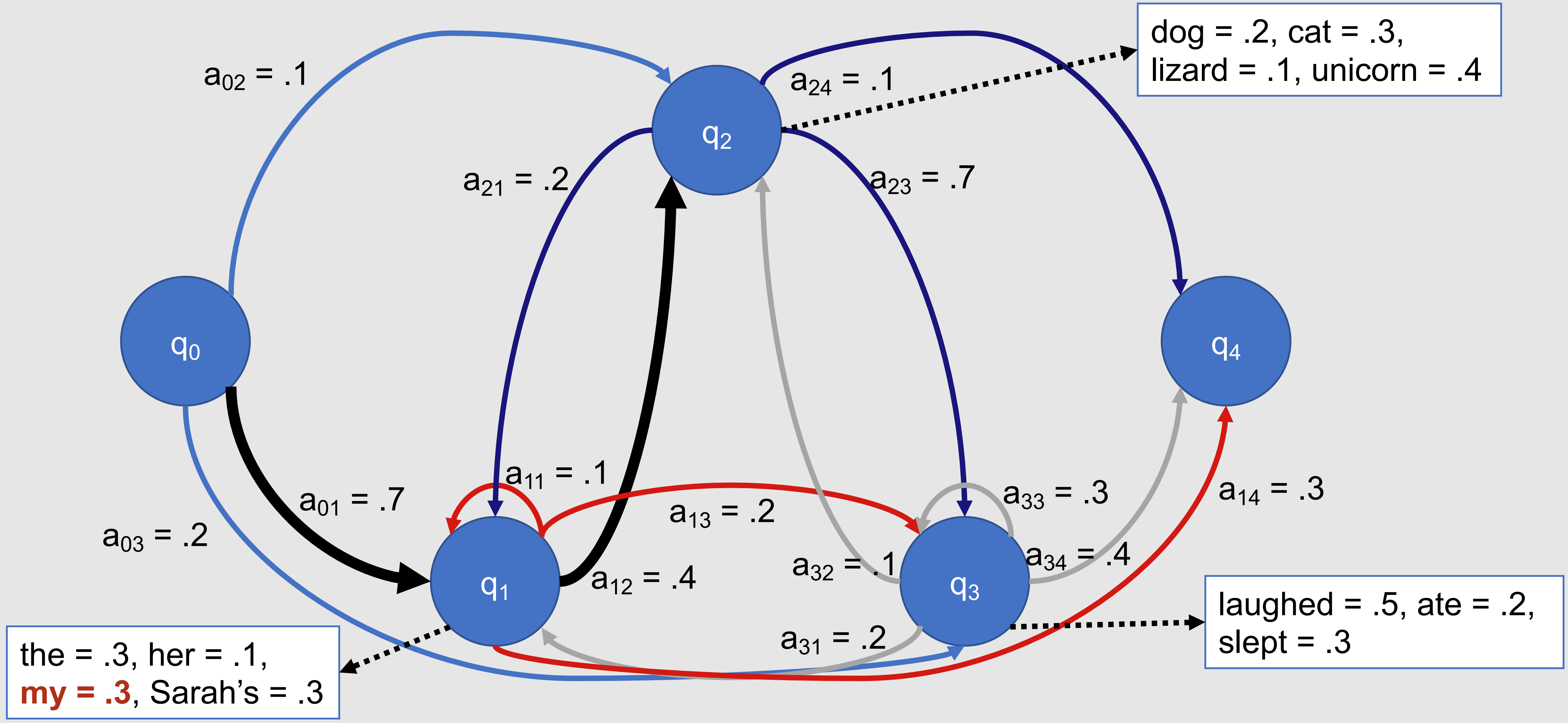
Sample Text Generation



Sample Text Generation



Sample Text Generation



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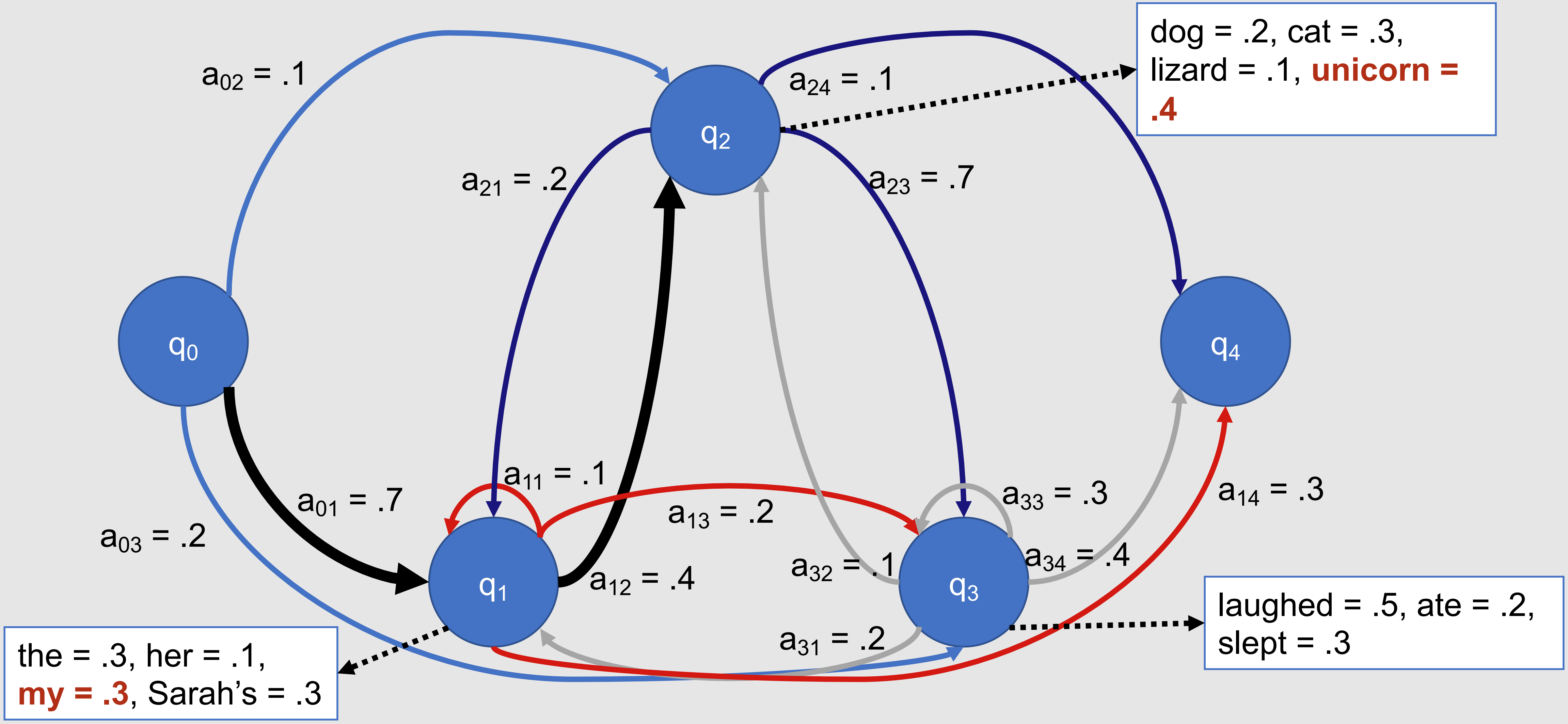

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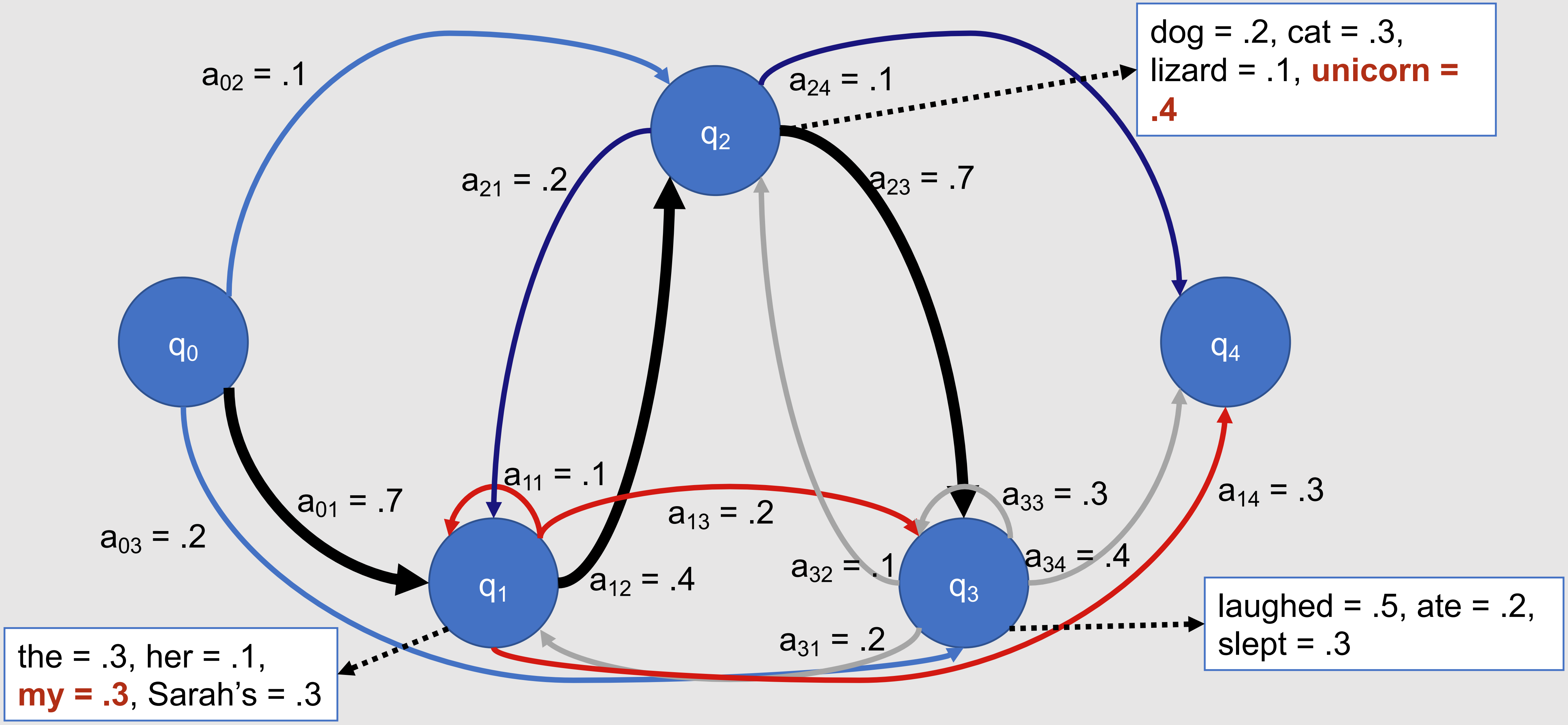

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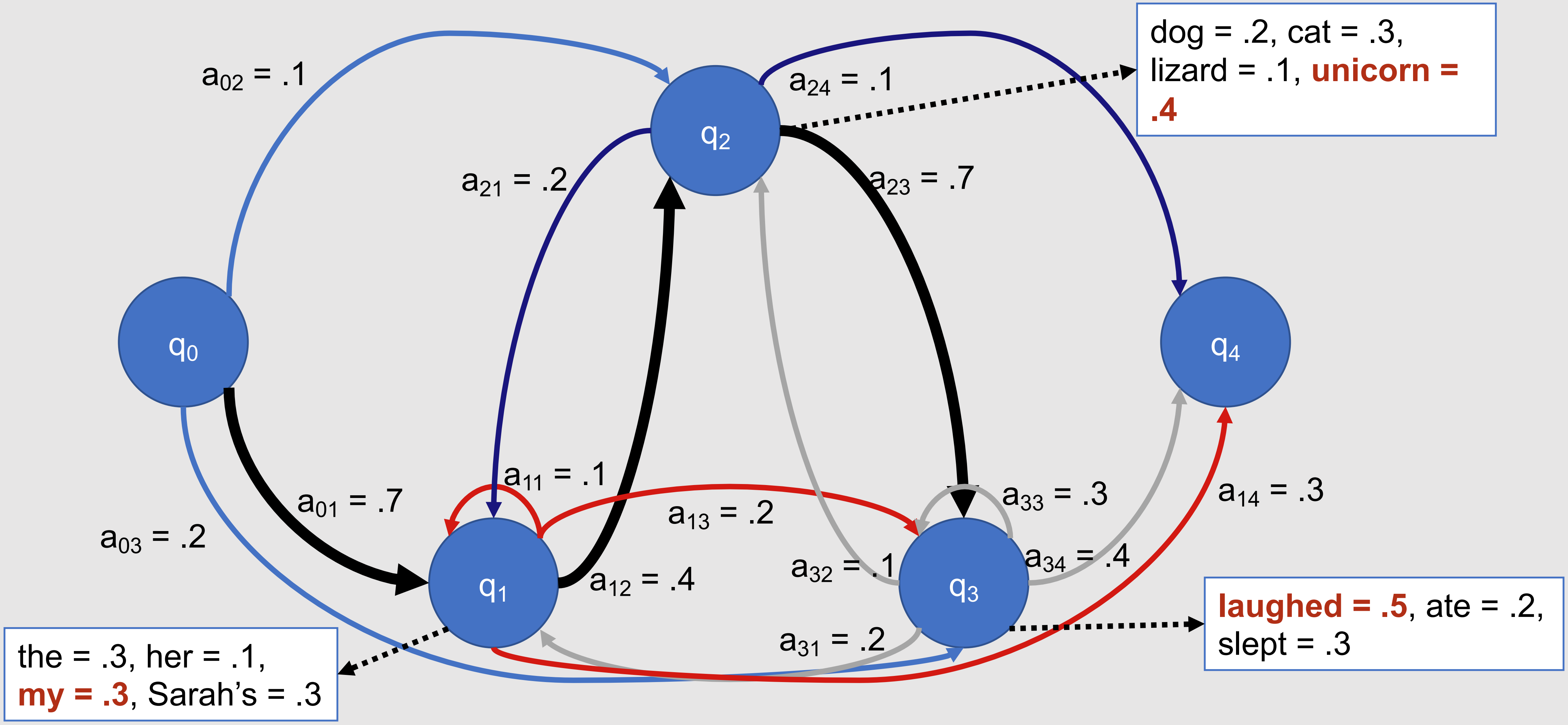

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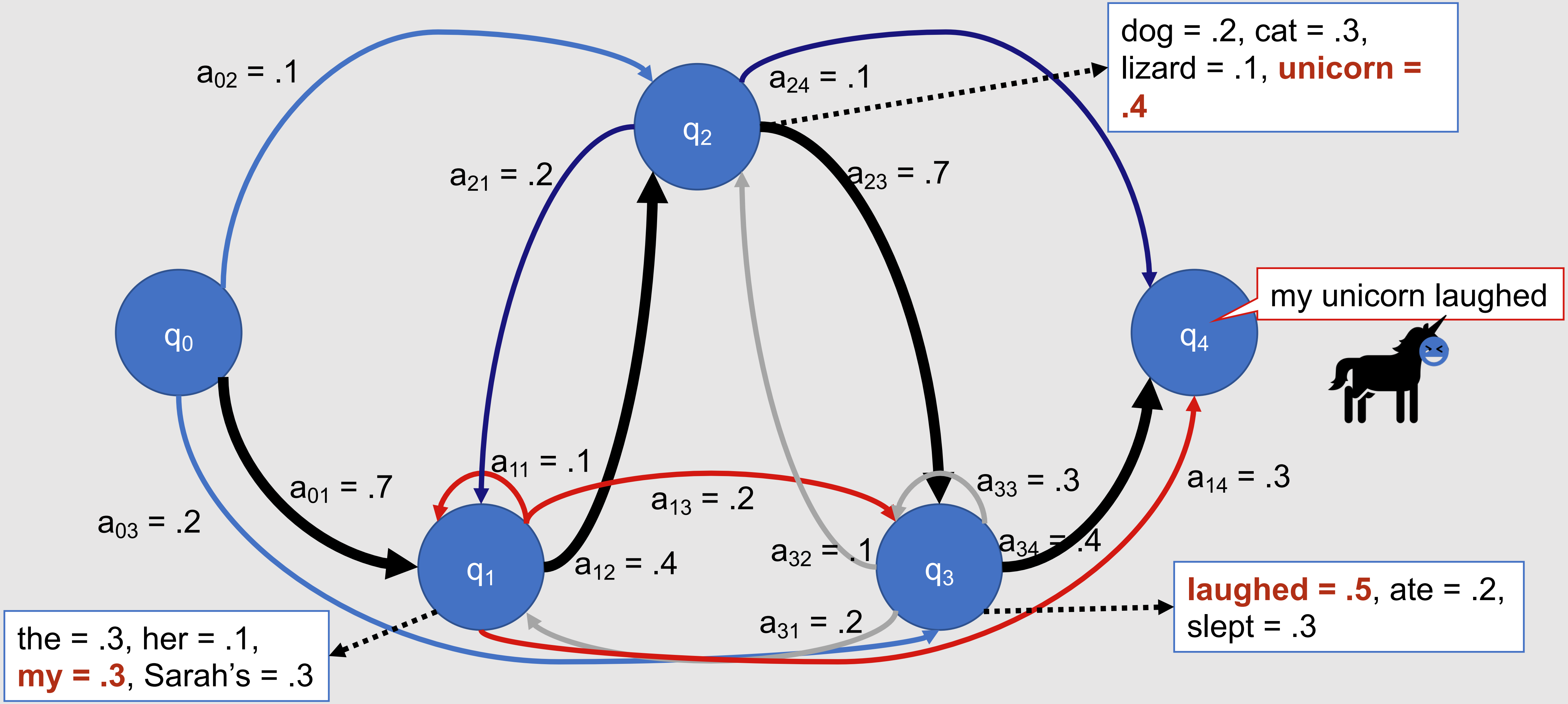


Three Fundamental HMM Problems

- Observation Likelihood: How likely is a particular observation sequence to occur?
- Decoding: What is the best sequence of hidden states for an observed sequence?
  - What is the best sequence of labels for our test data?
- Learning: What are the transition probabilities and observation likelihoods that best fit the observation sequence and HMM states?
  - How do we empirically fit our training data?

Observation Likelihood

- Given a sequence of observations and an HMM, what is the probability that this sequence was generated by the model?
- Allows the HMM to be used as a **language model**: A formal probabilistic model of a language that assigns a probability to each string by saying how likely that string was to have been generated by the language.
- Useful for two tasks:
  - Sequence classification
  - Selecting the most likely sequence

Sequence Classification

- Assuming an HMM is available for every possible class, what is the most likely class for a given observation sequence?
  - Which HMM is most likely to have generated the sequence?
- HMMs are commonly used in automated speech recognition (ASR) for this purpose
  - Given a set of sounds, what is the most likely word?

Most Likely Sequence

- Of two or more possible sequences, which one was most likely generated by a given HMM?
- Also useful for speech recognition
  - Rank alternative word sequence interpretations

How can we compute the observation likelihood?

- Naïve Solution:
  - Consider all possible state sequences, Q, of length T that the model, λ, could have traversed in generating the given observation sequence, O
  - Compute the probability of a given state sequence from A, and multiply it by the probability of generating the given observation sequence for that state sequence
    - $P(O,Q \mid \lambda) = P(O \mid Q, \lambda) * P(Q \mid \lambda)$
  - Repeat for all possible state sequences, and sum over all to get $P(O \mid \lambda)$
- But, this is computationally complex!
  - $O(TN^T)$

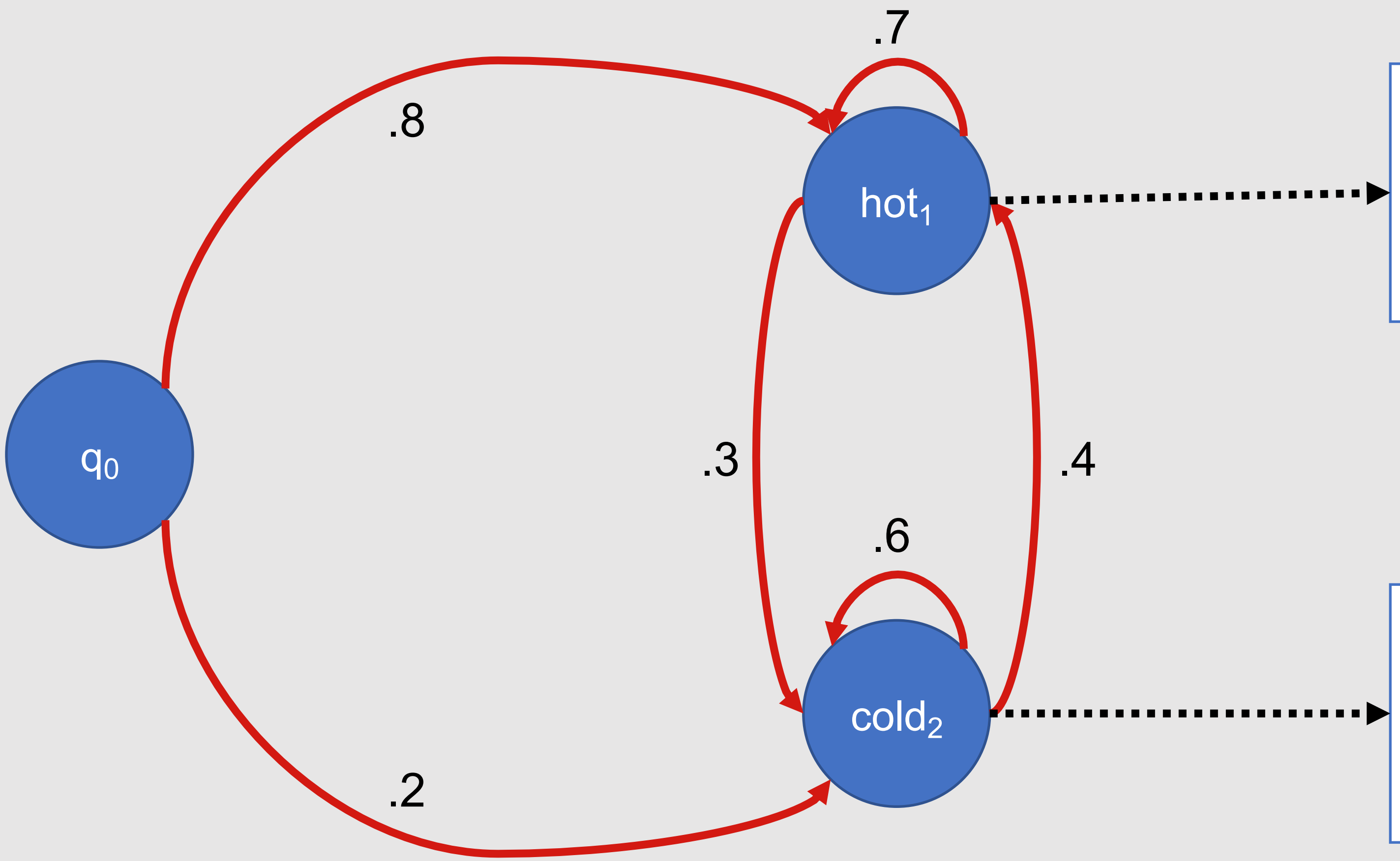
How can we compute the observation likelihood?

- Efficient Solution:
  - **Forward Algorithm:** Dynamic programming algorithm that computes the observation probability by summing over the probabilities of all possible hidden state paths that could generate the observation sequence.
  - Implicitly folds each of these paths into a single forward trellis
- Why does this work?
  - Markov assumption (the probability of being in any state at a given time *t* only relies on the probability of being in each possible state at time *t*-1).
- Works in $O(TN^2)$ time!

Sample Problem

- It is 2799 and you are a climatologist studying the history of global warming
- Unfortunately, you have no records of the weather in Baltimore for the summer of 2007, although you do know how likely it was in general to move from a hot day to a cold day and so forth at that time
- Fortunately, a major breakthrough occurs: you find Jason Eisner's diary, which lists how many ice cream cones he ate every day that summer
- You decide to use those observations to estimate whether each day in a three-day sequence was hot or cold
    - Day 1: 3 ice cream cones
    - Day 2: 1 ice cream cone
    - Day 3: 3 ice cream cones

Corresponding HMM



B1

$$\begin{bmatrix} P(1|hot_1) \\ P(2|hot_1) \\ P(3|hot_1) \end{bmatrix} = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix}$$

B2

$$\begin{bmatrix} P(1|cold_1) \\ P(2|cold_1) \\ P(3|cold_1) \end{bmatrix} = \begin{bmatrix} .5 \\ .4 \\ .1 \end{bmatrix}$$

How do you compute your forward probabilities?

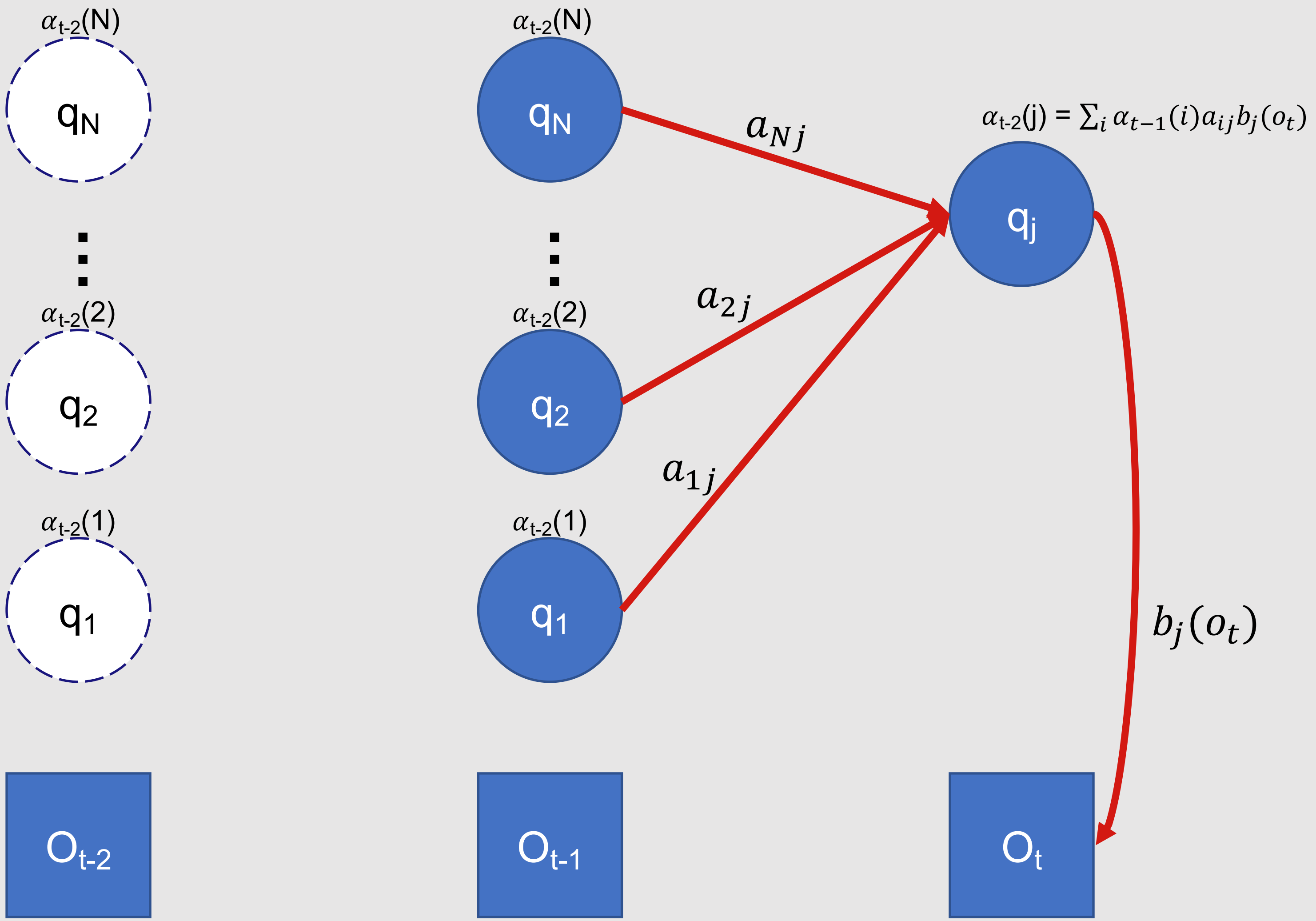
- Let $\alpha_i(j)$ be the probability of being in state *j* after seeing the first *t* observations, given your HMM λ
- $\alpha_i(j)$ is computed by summing over the probabilities of every path that could lead you to this cell
  - $\alpha_i(j) = P(o_1, o_2 \ldots o_t, q_t = j|\lambda) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t)$
    - $q_t = j$ is the probability that the *t*^th^ state in the sequence of states is state *j*
  - $\alpha_{t-1}(i)$: The previous forward path probability from the previous time step
  - a_{ij}: The transition probability from previous state q_i to current state q_j
  - $b_j(o_t)$: The state observation likelihood of the observed item o_t given the current state *j*

Formal Algorithm

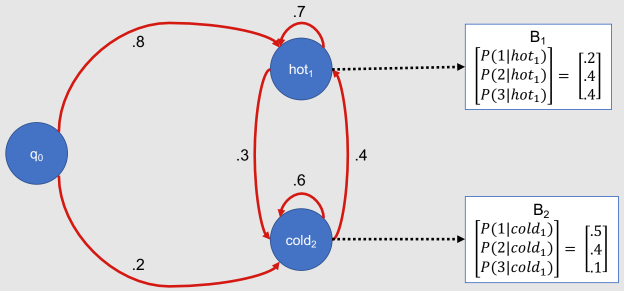
```
create a probability matrix forward[N+2,T]

for each state q in [1, …, N] do:
    forward[q,1] ← a_{0,q} * b_q(o_1)
for each time step t from 2 to T do:
    for each state q from 1 to N do:
        forward[q,t] ←
```
$\sum_{q'=1}^{N} forward[q',t-1] * a_{q',q} * b_s(o_t)$

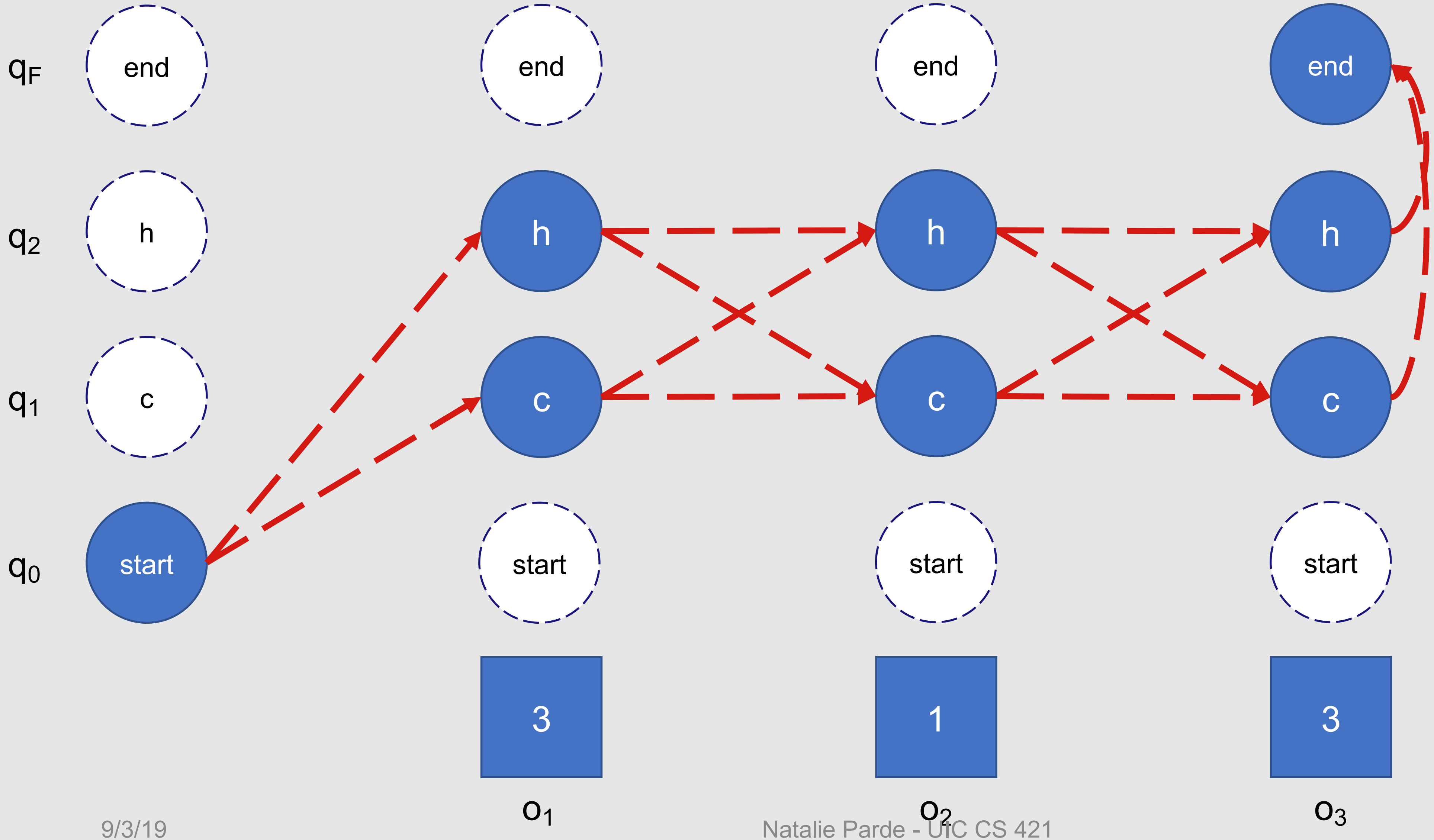
`forward[q_F,T] ←` $\sum_{q=1}^{N} forward[q,T] * a_{s,q_F}$

Forward Step



Forward Trellis





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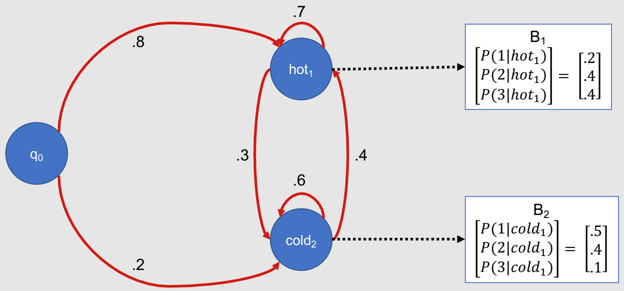
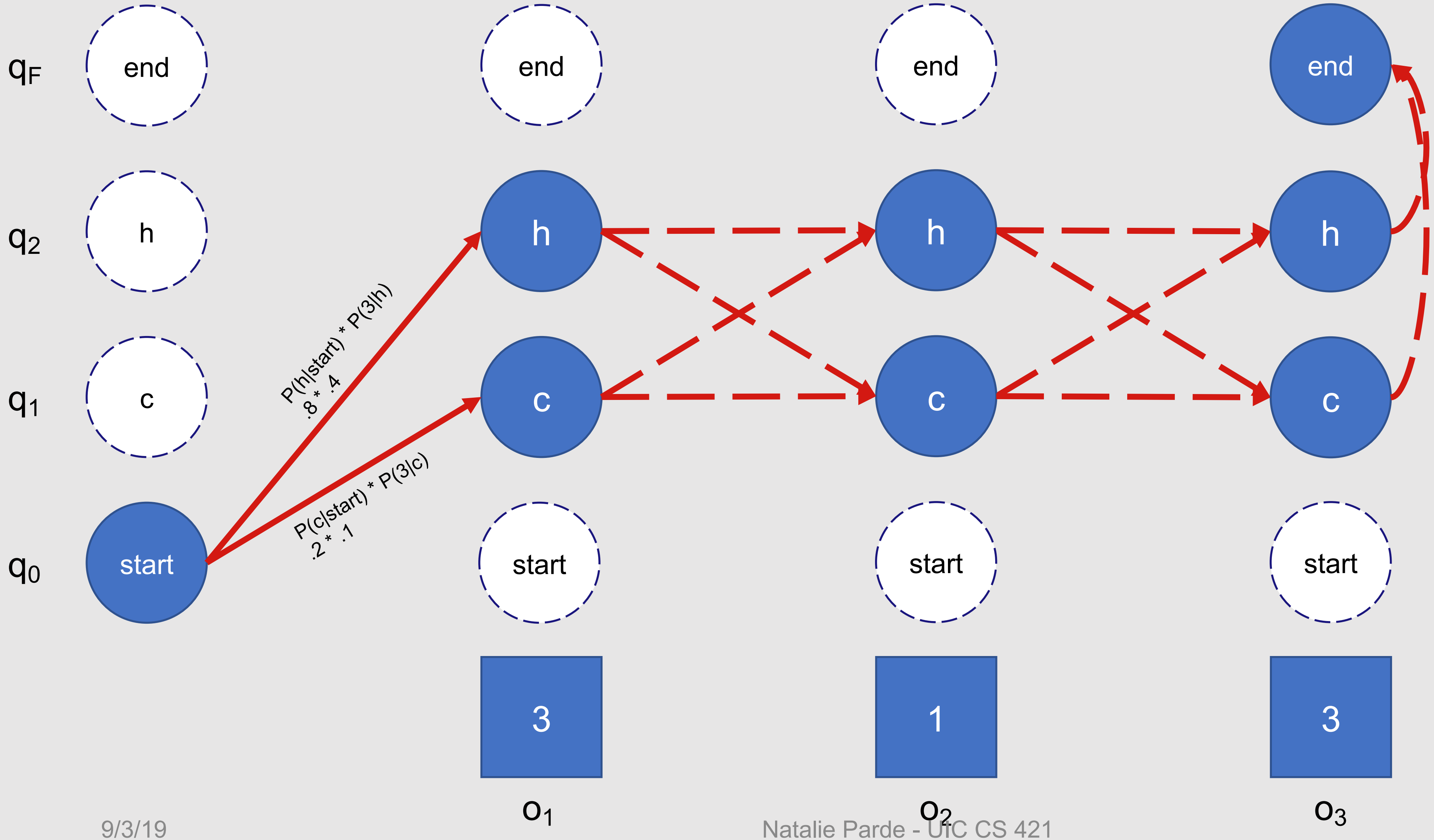
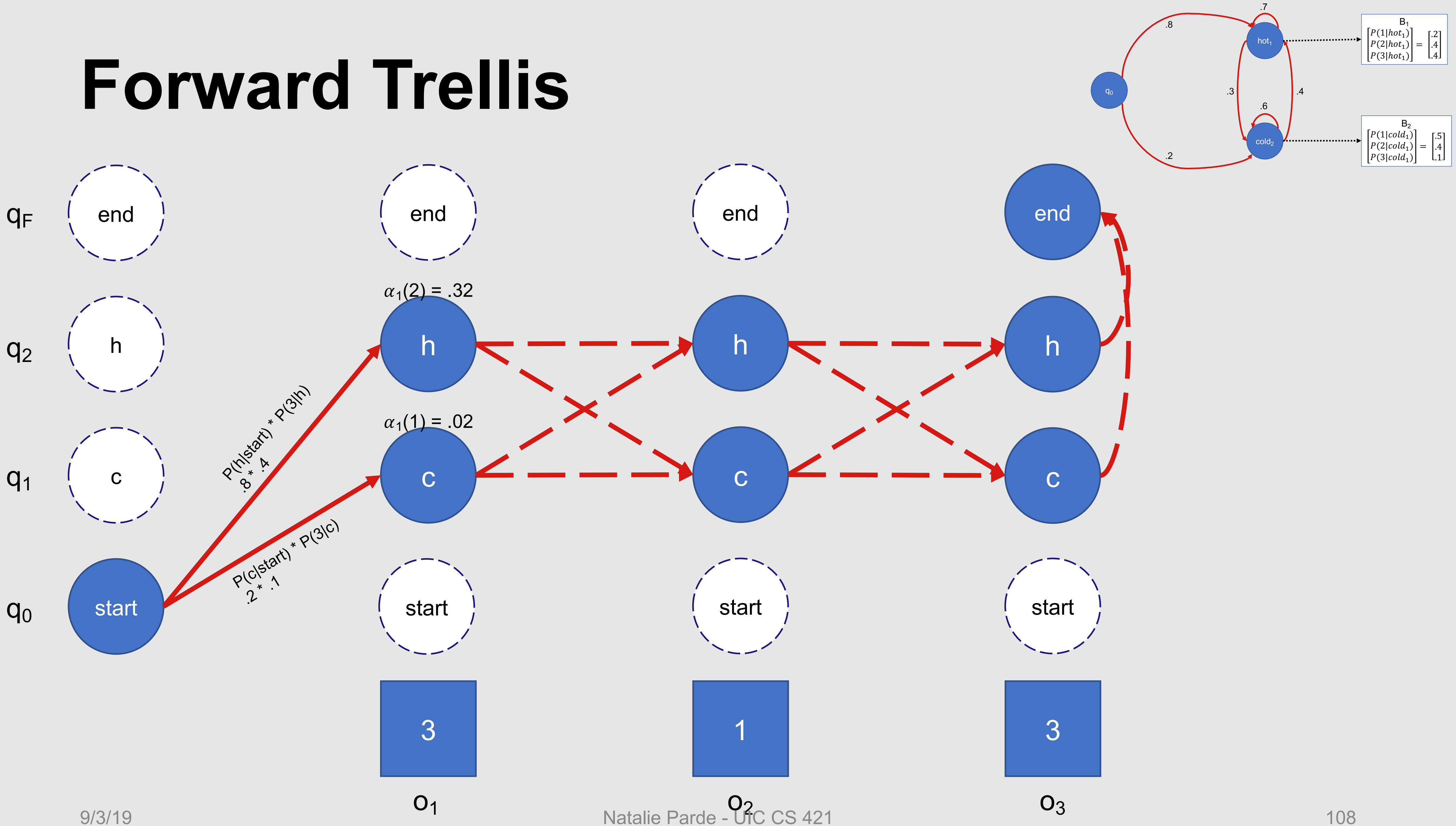




Forward Trellis



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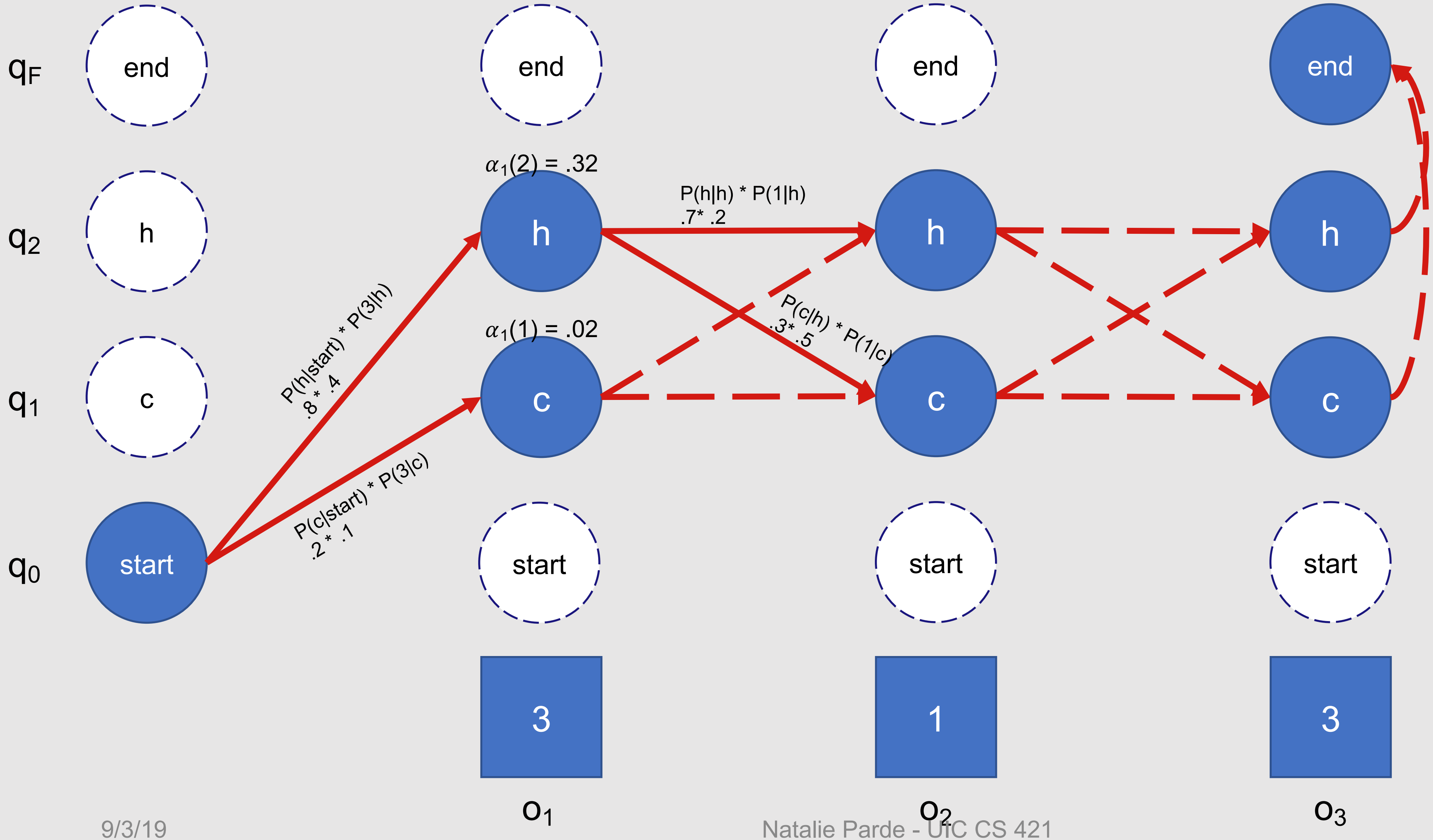

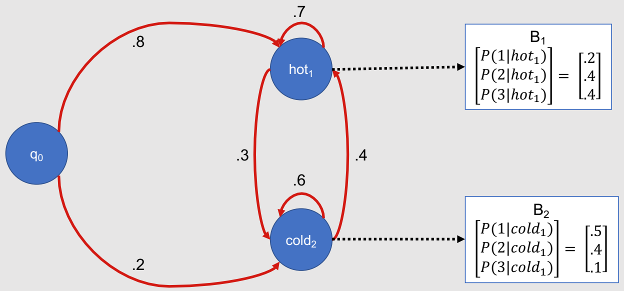

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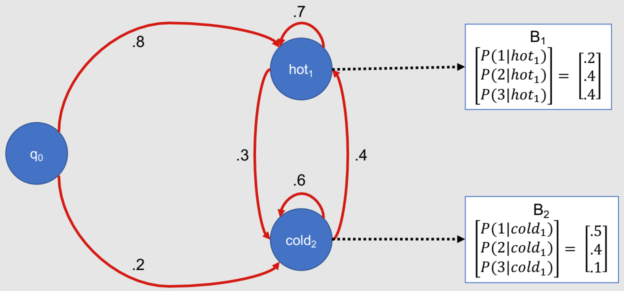

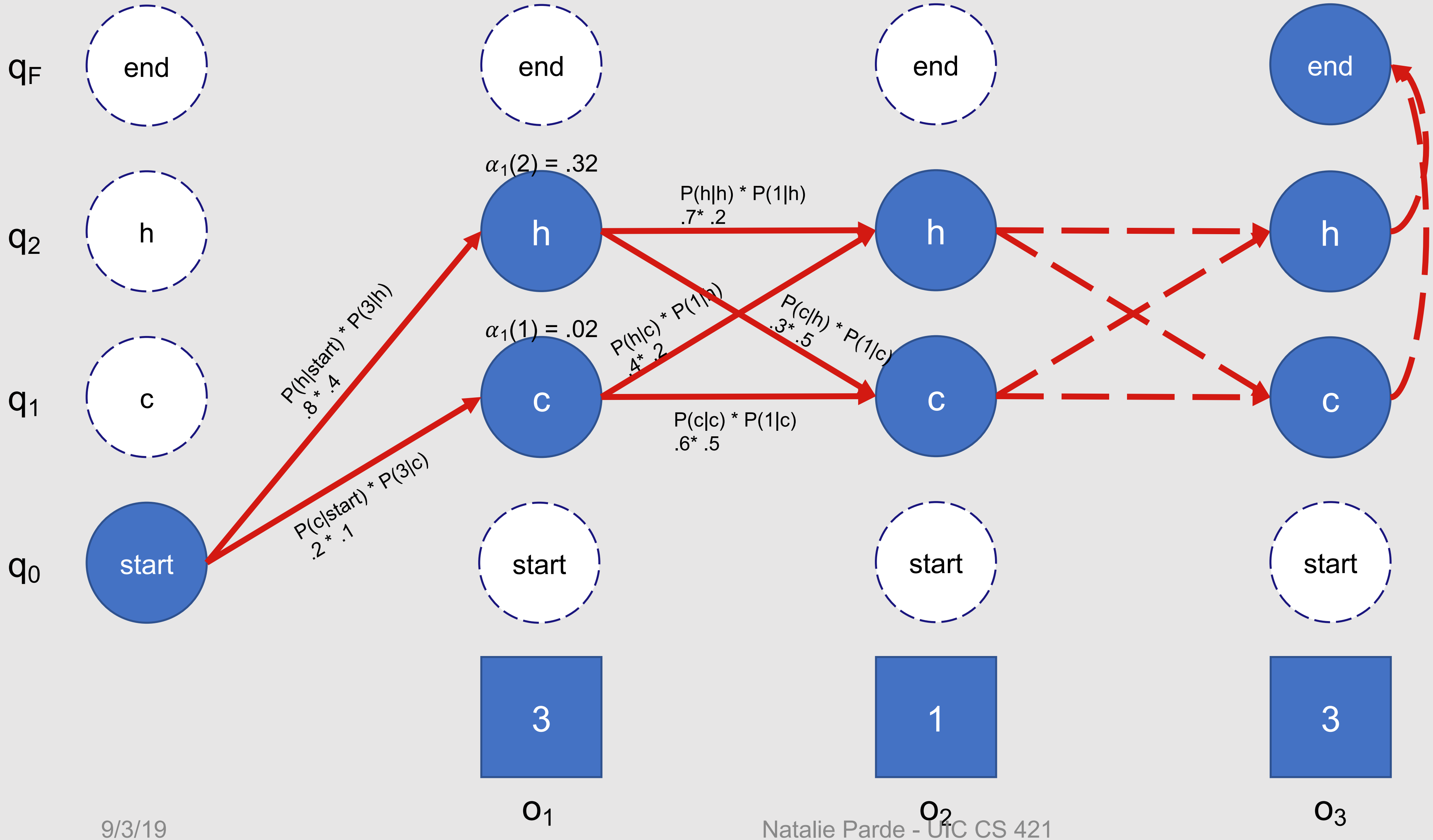

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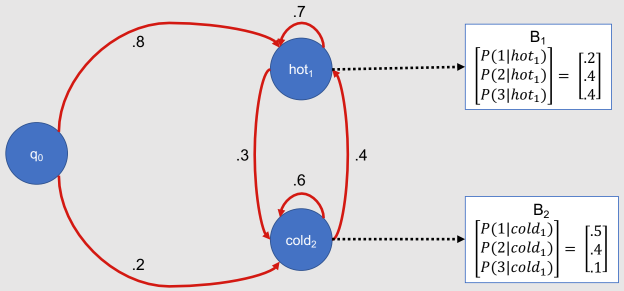
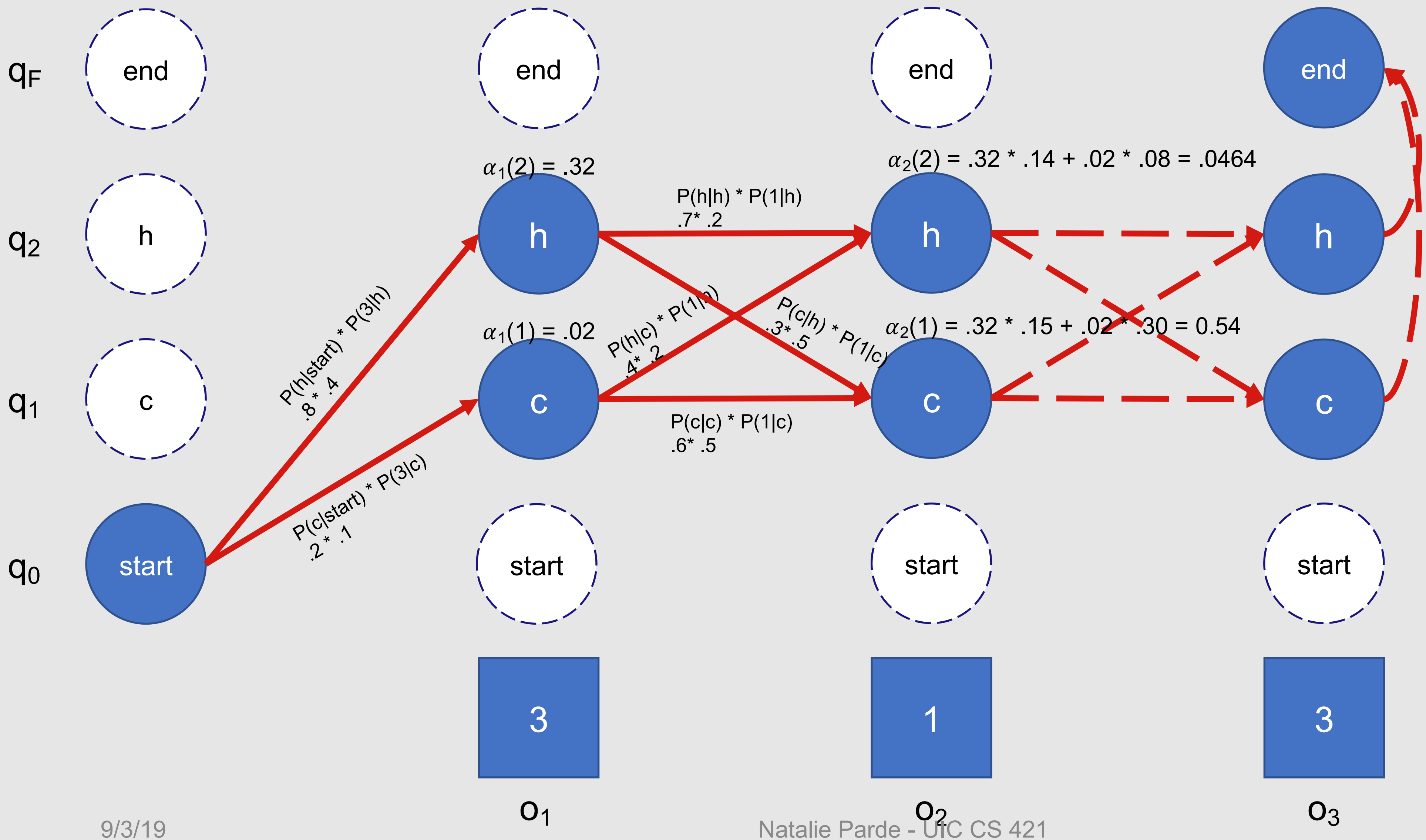
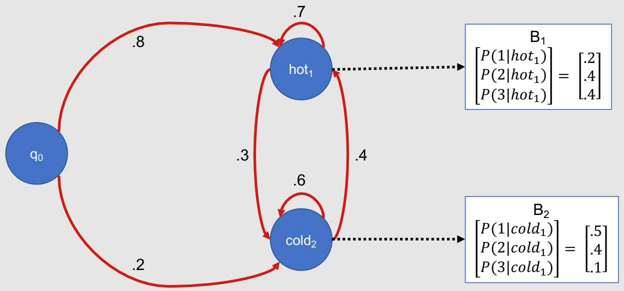
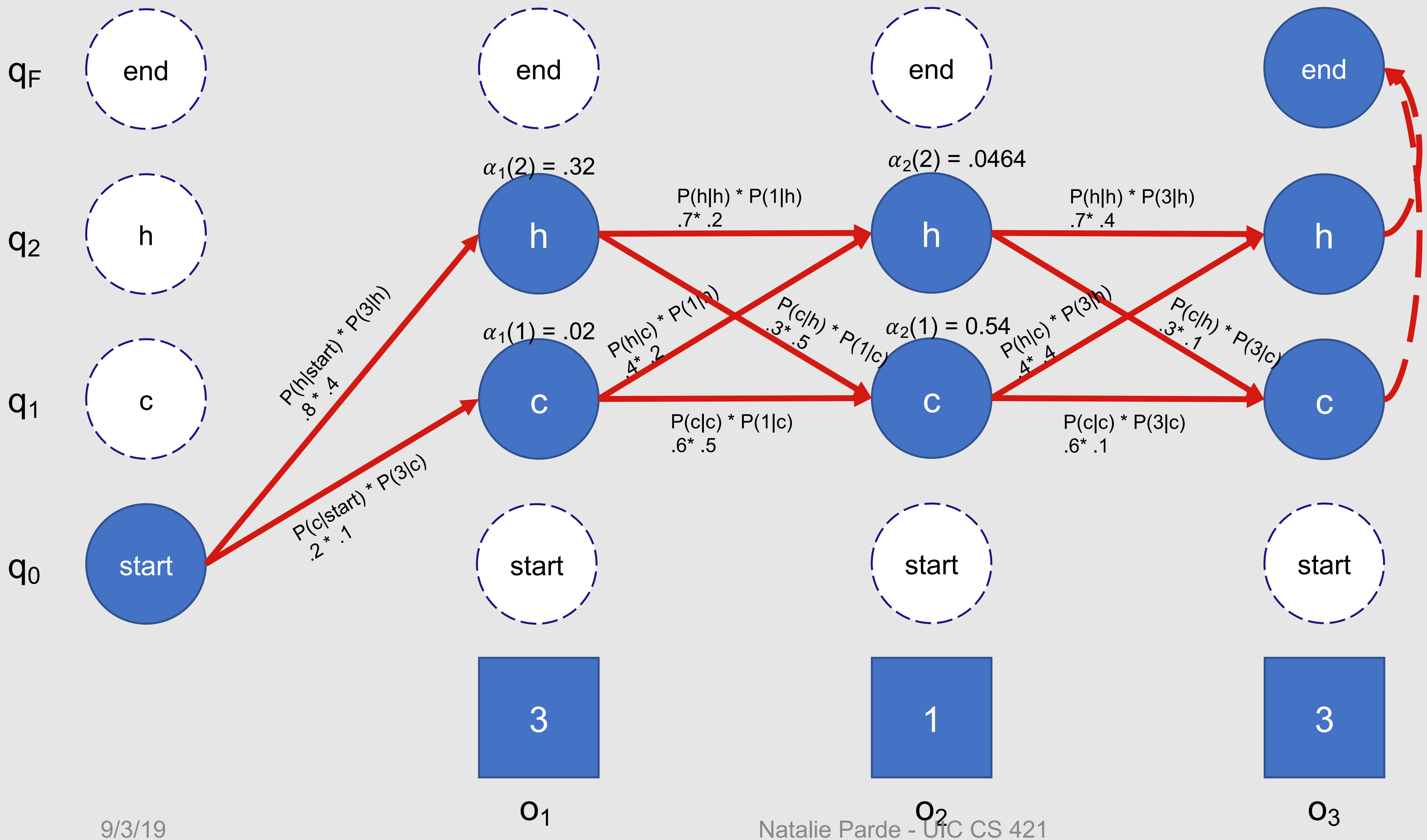


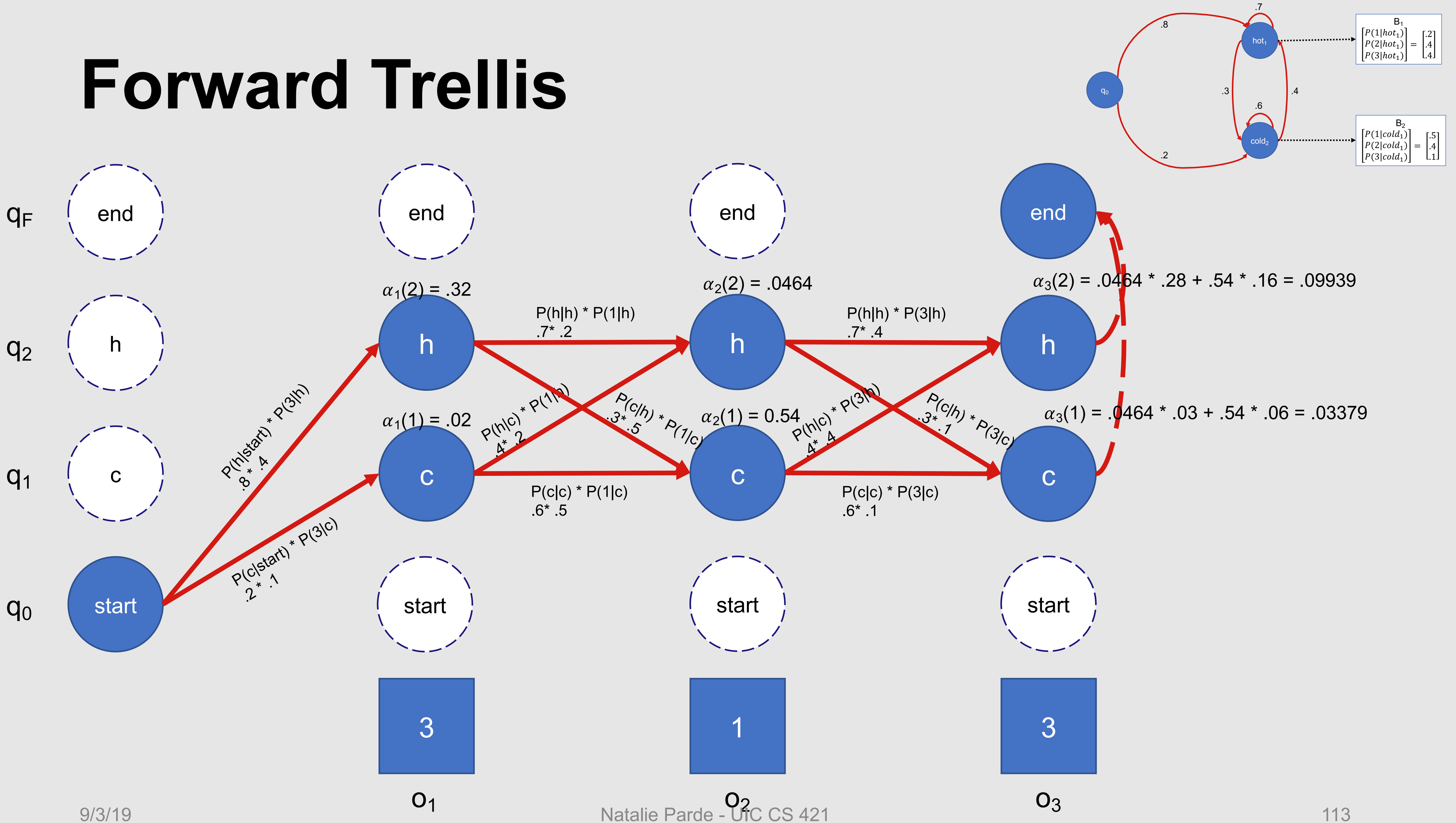


Forward Trellis





Forward Trellis



Decoding

- Given an observation sequence and an HMM, what is the best hidden state sequence?
  - How do we choose a state sequence that is optimal in some sense (e.g., best explains the observations)?
- Very useful for sequence labeling!

Naïve Approach:

- For each hidden state sequence Q, compute P(O|Q)
- Pick the sequence with the highest probability

However, this is computationally inefficient!

- $O(N^T)$

Decoding

How can we decode sequences more efficiently?

- **Viterbi Algorithm**
  - Another dynamic programming algorithm
  - Uses a similar trellis to the Forward algorithm
- Viterbi time complexity: $O(N^2T)$

Viterbi Intuition

- **Goal:** Compute the joint probability of the observation sequence together with the best state sequence
- So, **recursively compute the probability of the most likely subsequence of states** that accounts for the first *t* observations and ends in state q_j.
  - $v_t(j) = \max_{q_0, q_1, \ldots, q_{t-1}} P\big(q_0, q_1, \ldots, q_{t-1}, o_1, \ldots, o_t, q_t = q_j | \lambda\big)$
- Also **record backpointers** that subsequently allow you to backtrace the most probable state sequence
  - $bt_t(j)$ stores the state at time *t*-1 that maximizes the probability that the system was in state q_j at time *t*, given the observed sequence

Formal Algorithm

```
create a path probability matrix Viterbi[N+2,T]

for each state q in [1,…,N] do:
        Viterbi[q,1] ← a_{0,q} * b_q(o_1)
        backpointer[q,1] ← 0
for each time step t in [2,…,T] do:
        for each state q in [1,…,N] do:
```

$$viterbi[q,t] \leftarrow \max_{q' \in [1,\ldots,N]} viterbi[q',t-1] * a_{q',q} * b_q(o_t)$$

$$backpointer[q,t] \leftarrow \operatorname{argmax}_{q' \in [1,\ldots,N]} viterbi[q',t-1] * a_{q',q}$$

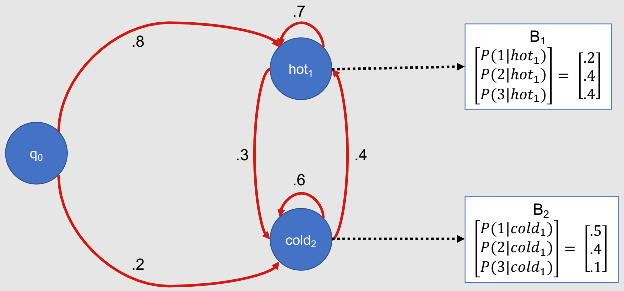
$$viterbi[q_F,T] \leftarrow \max_{q' \in [1,\ldots,N]} viterbi[q,T] * a_{q,q_F}$$

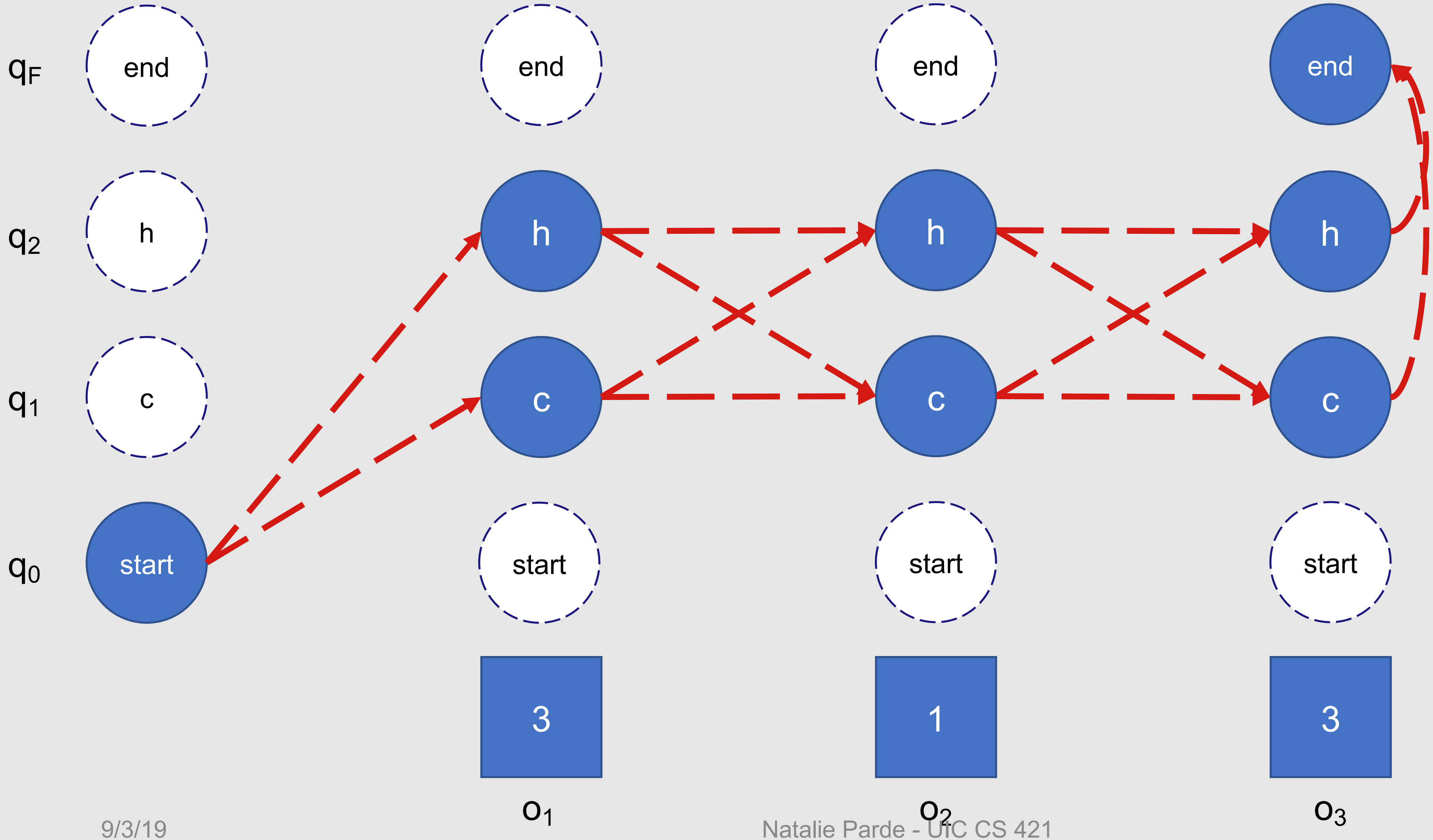
$$backpointer[q_F,T] \leftarrow \operatorname{argmax}_{q' \in [1,\ldots,N]} viterbi[q,T] * a_{q,q_F}$$

Seem familiar?

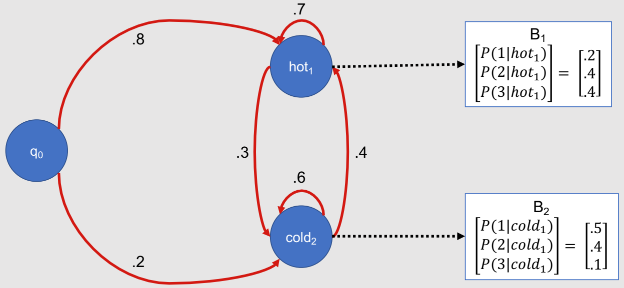
- Viterbi is basically the forward algorithm + backpointers, and substituting a max function for the summation operator

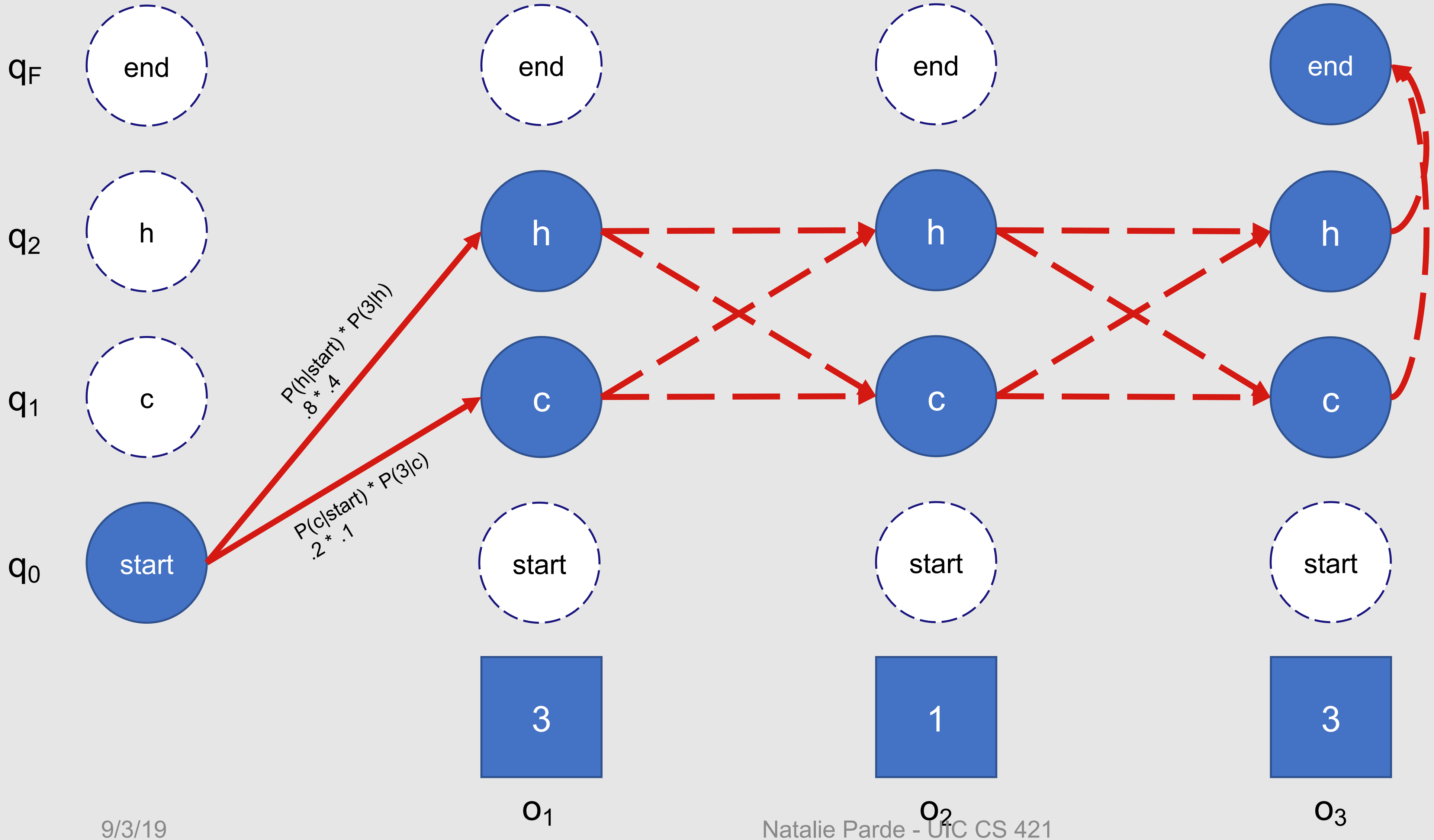
Viterbi Trellis



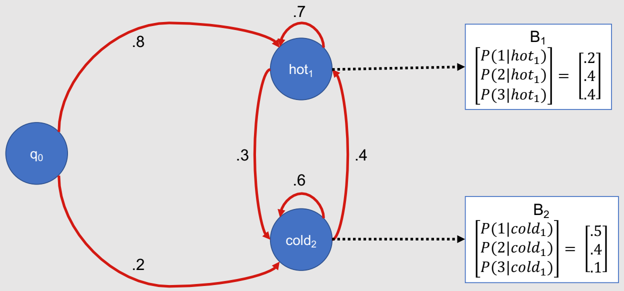


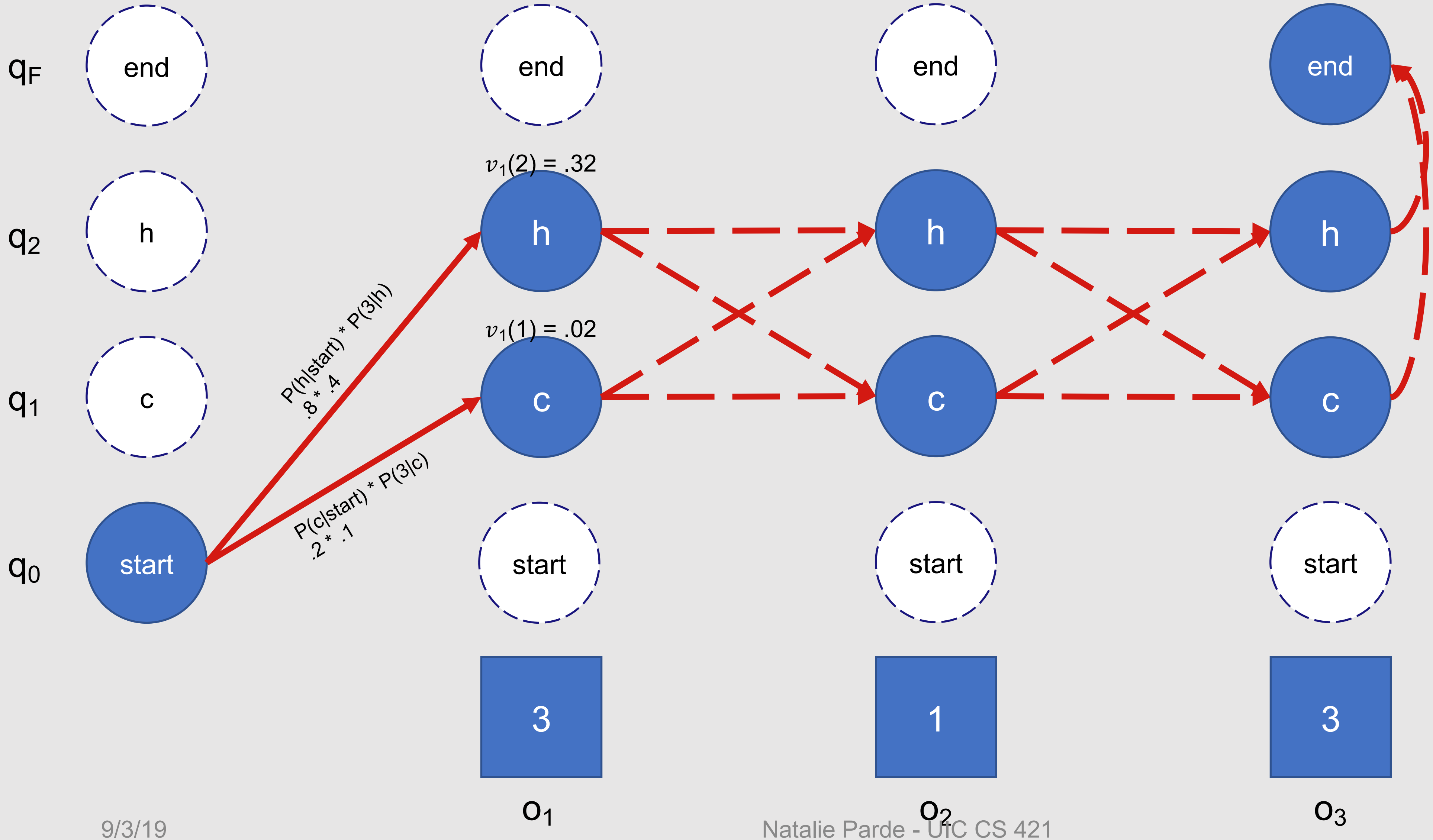
Viterbi Trellis





Viterbi Trellis





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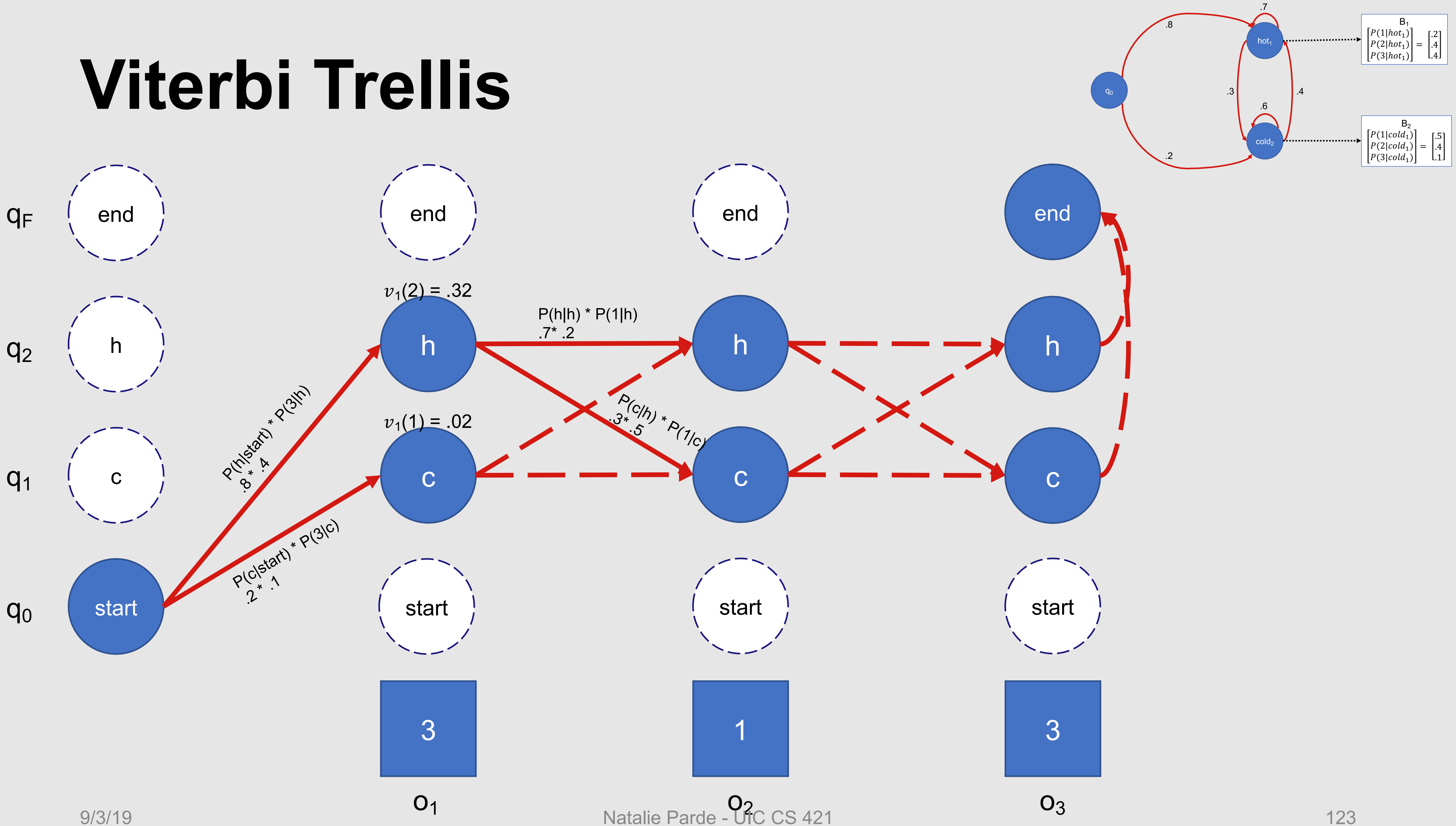

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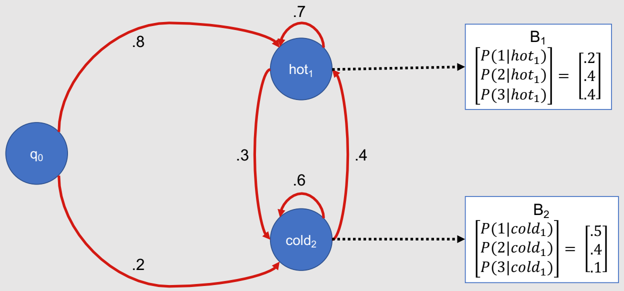

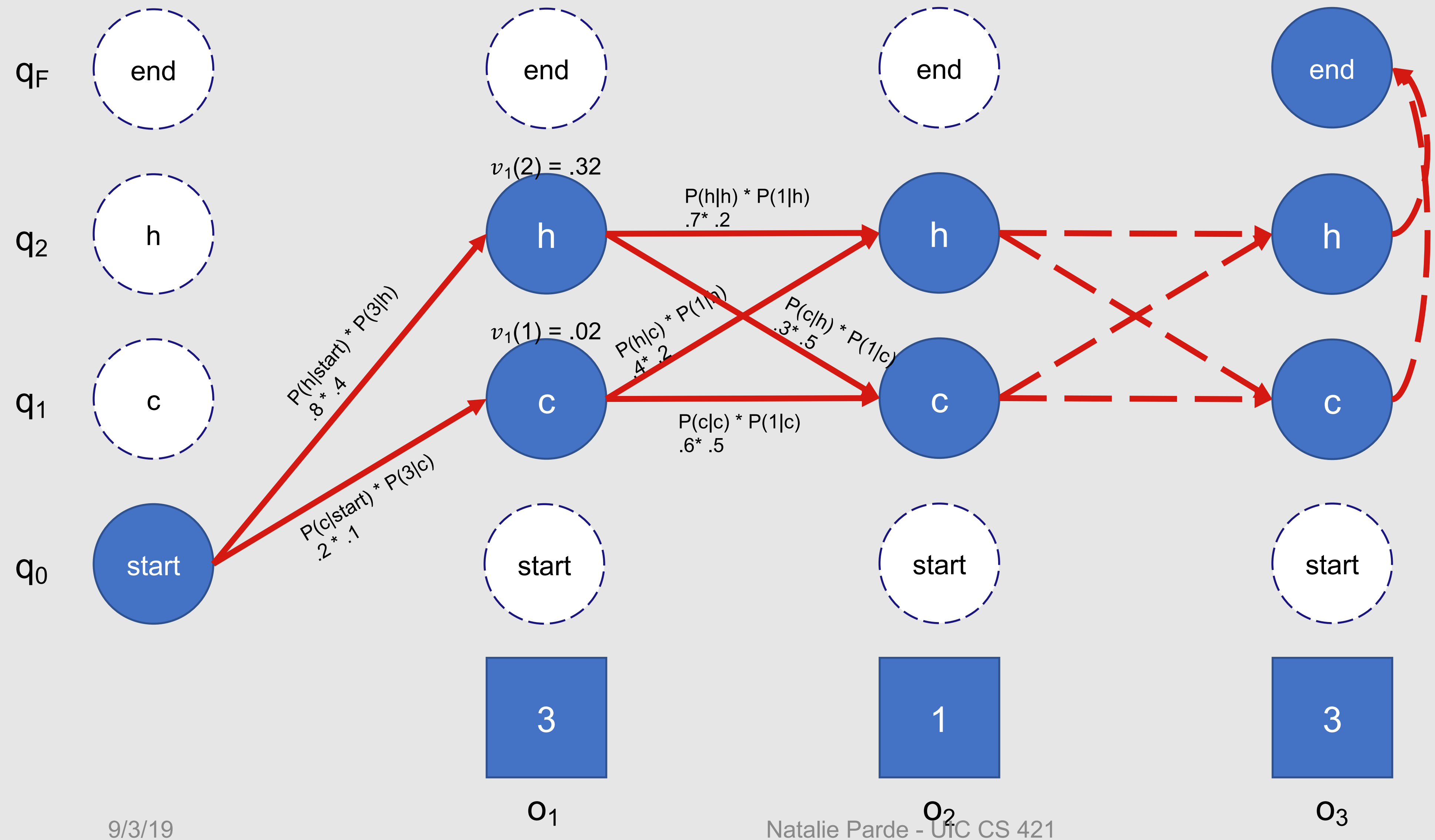

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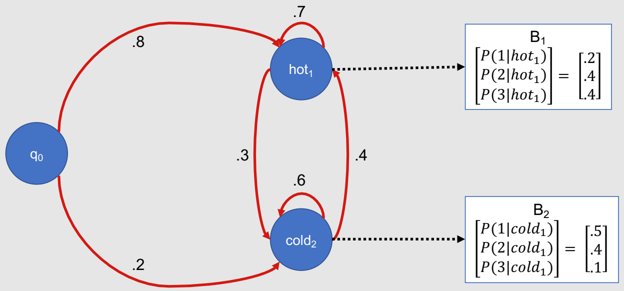

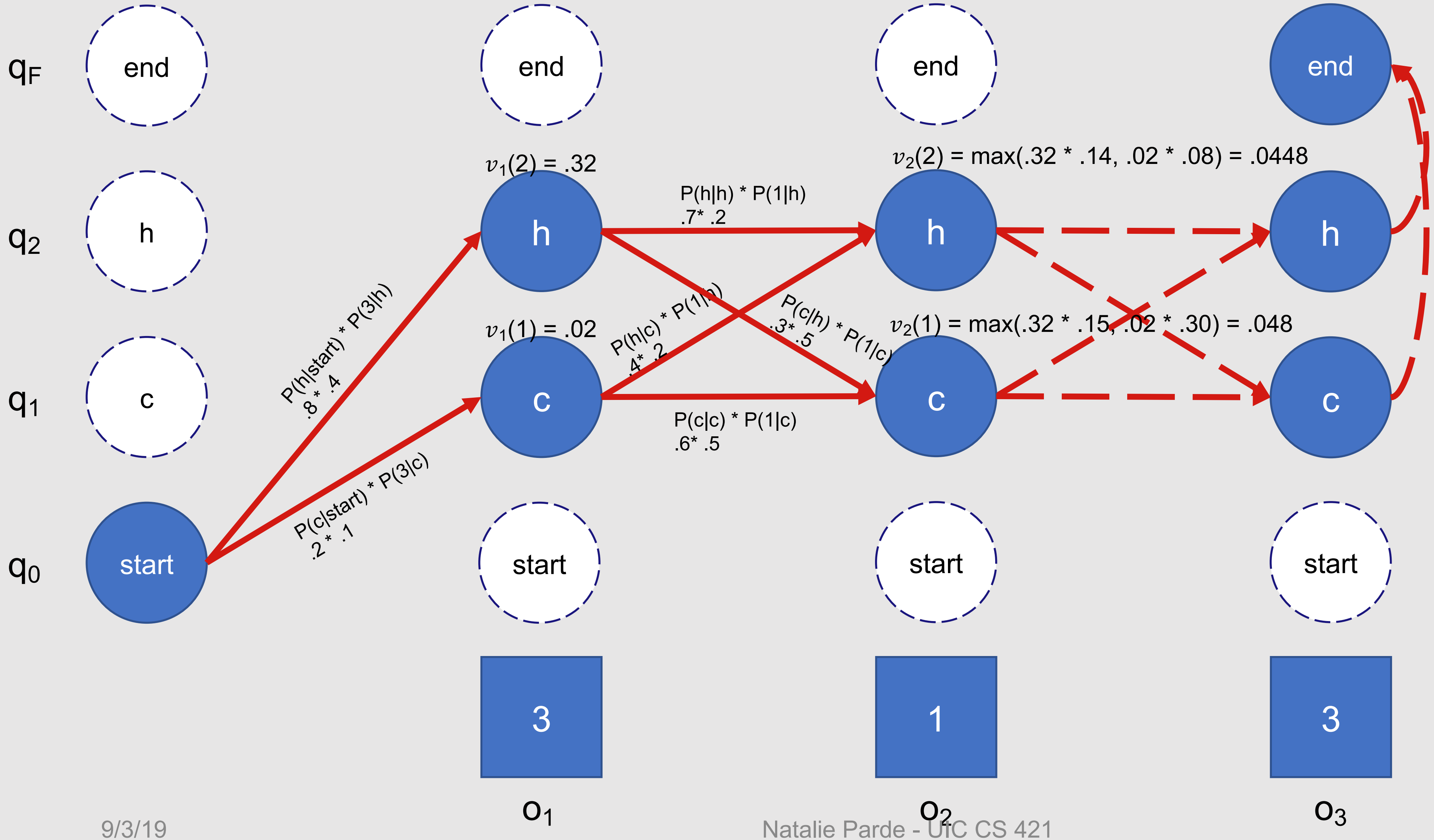

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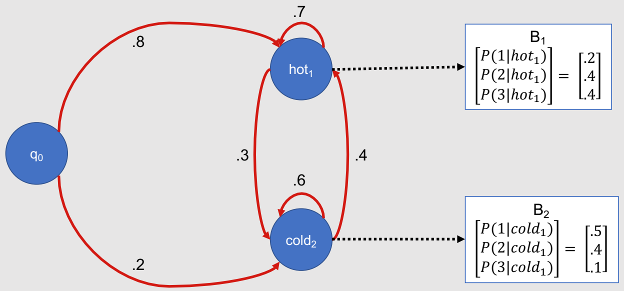

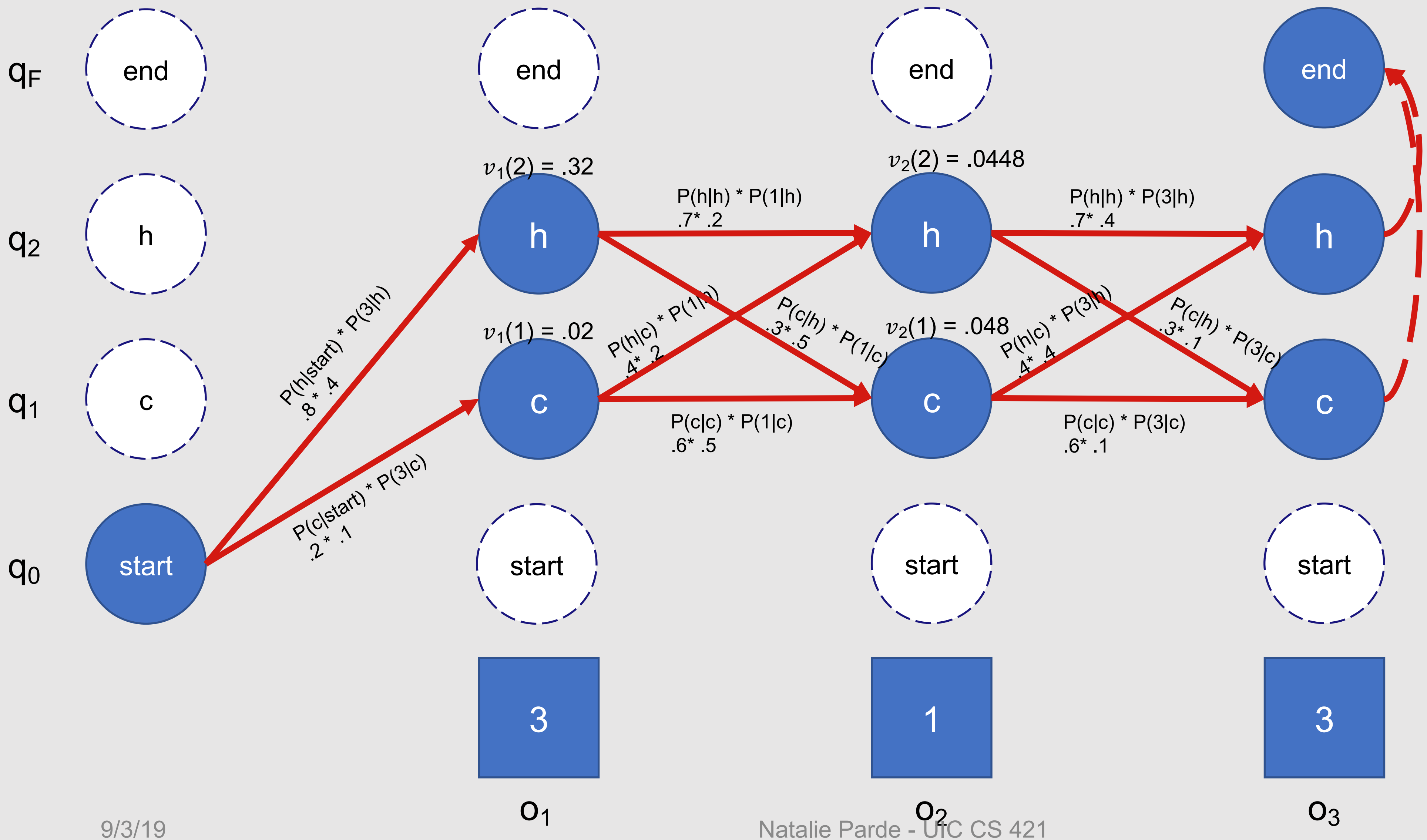

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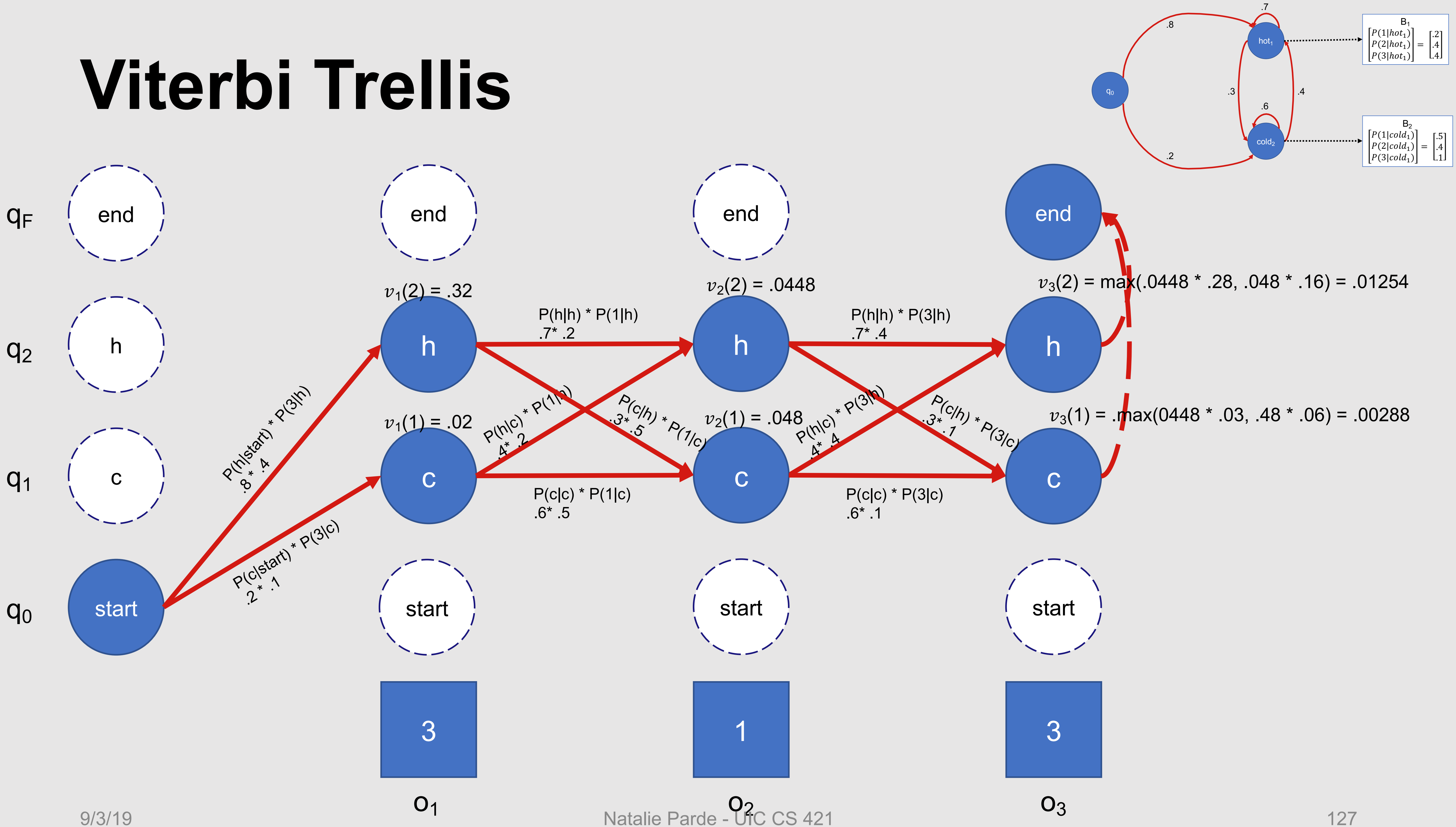
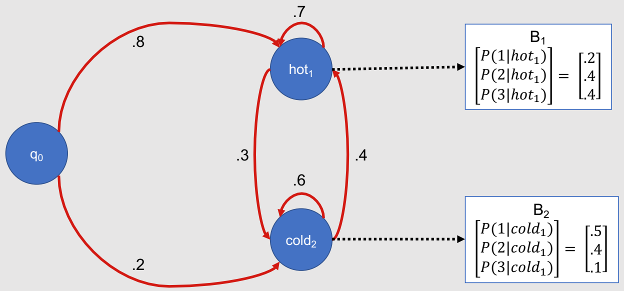
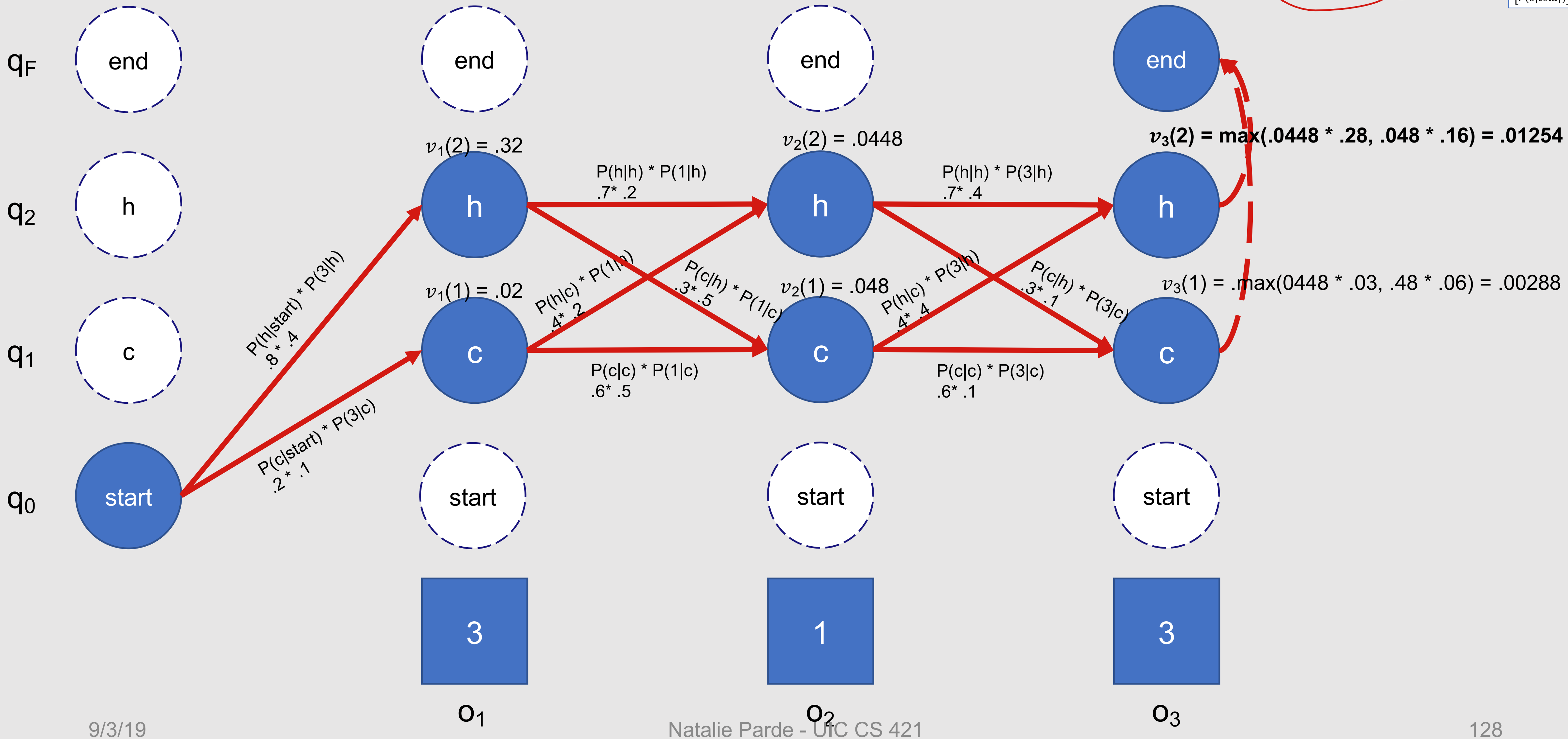


Viterbi Backtrace





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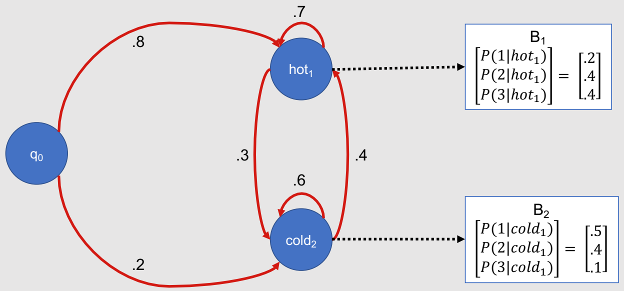

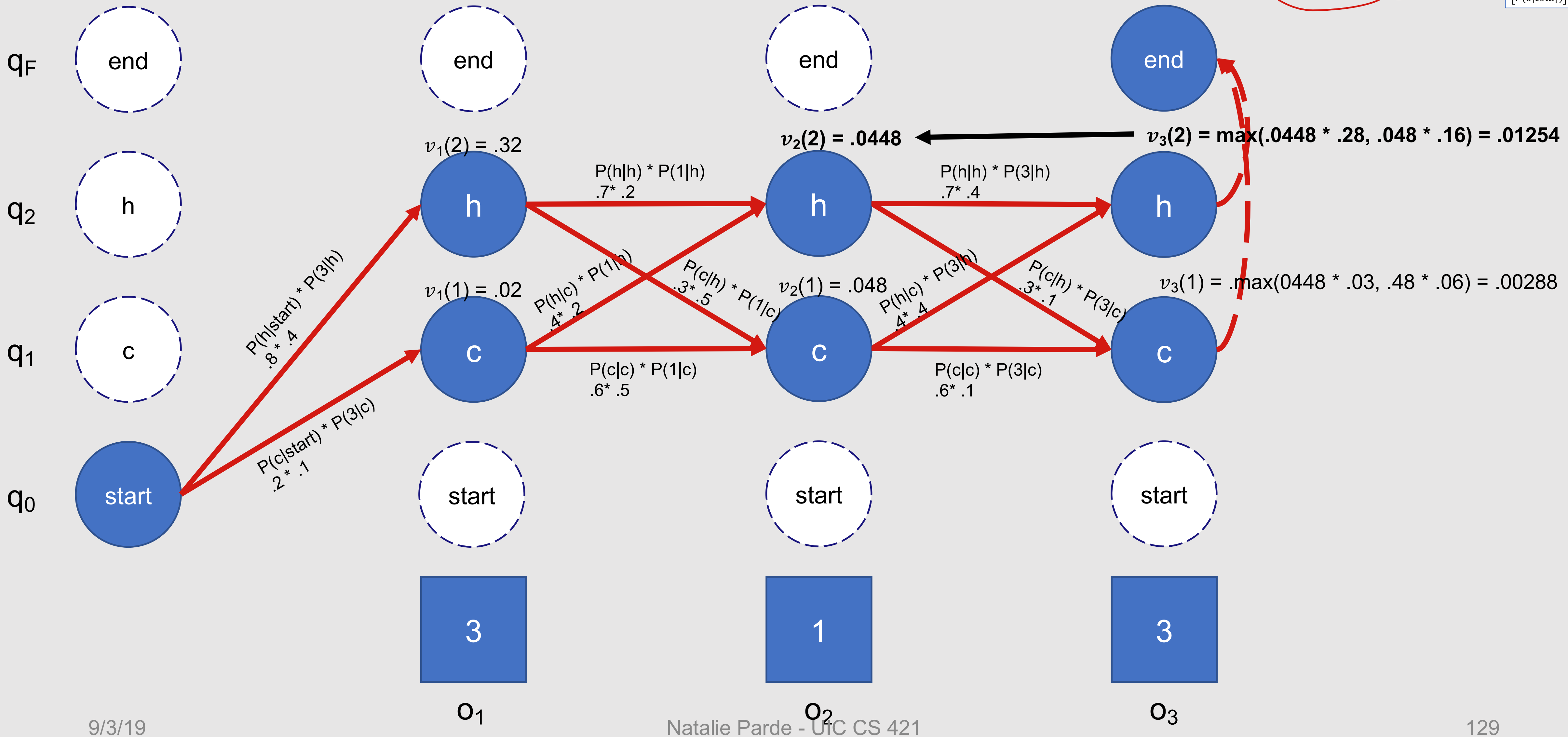

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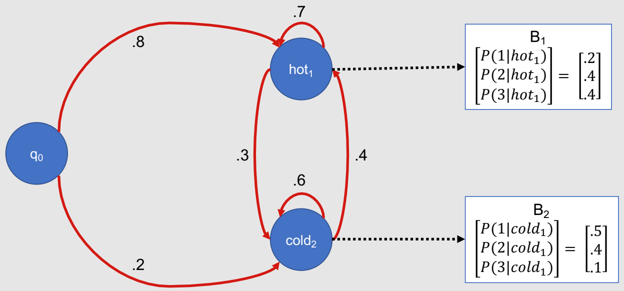

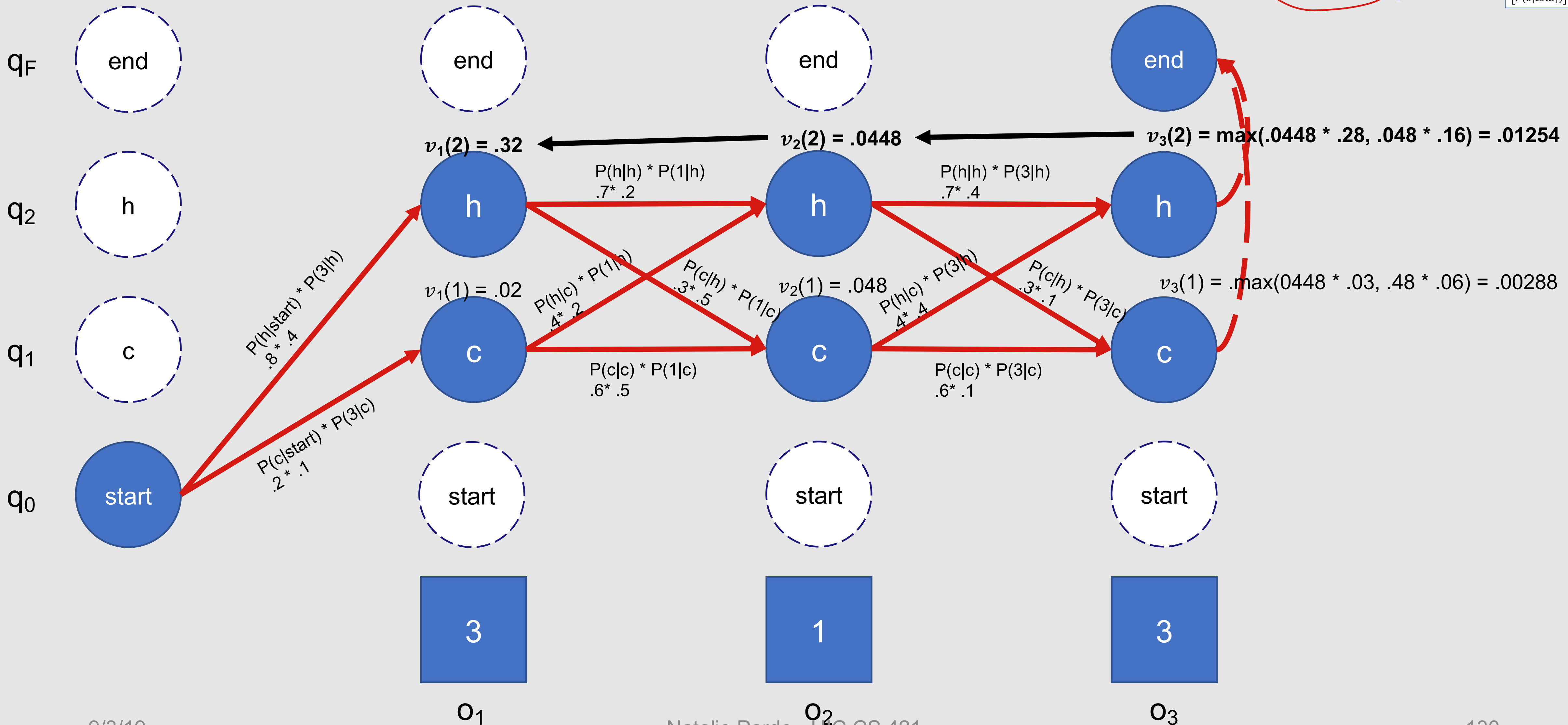

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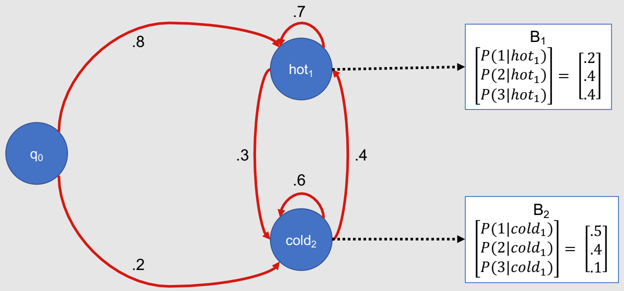
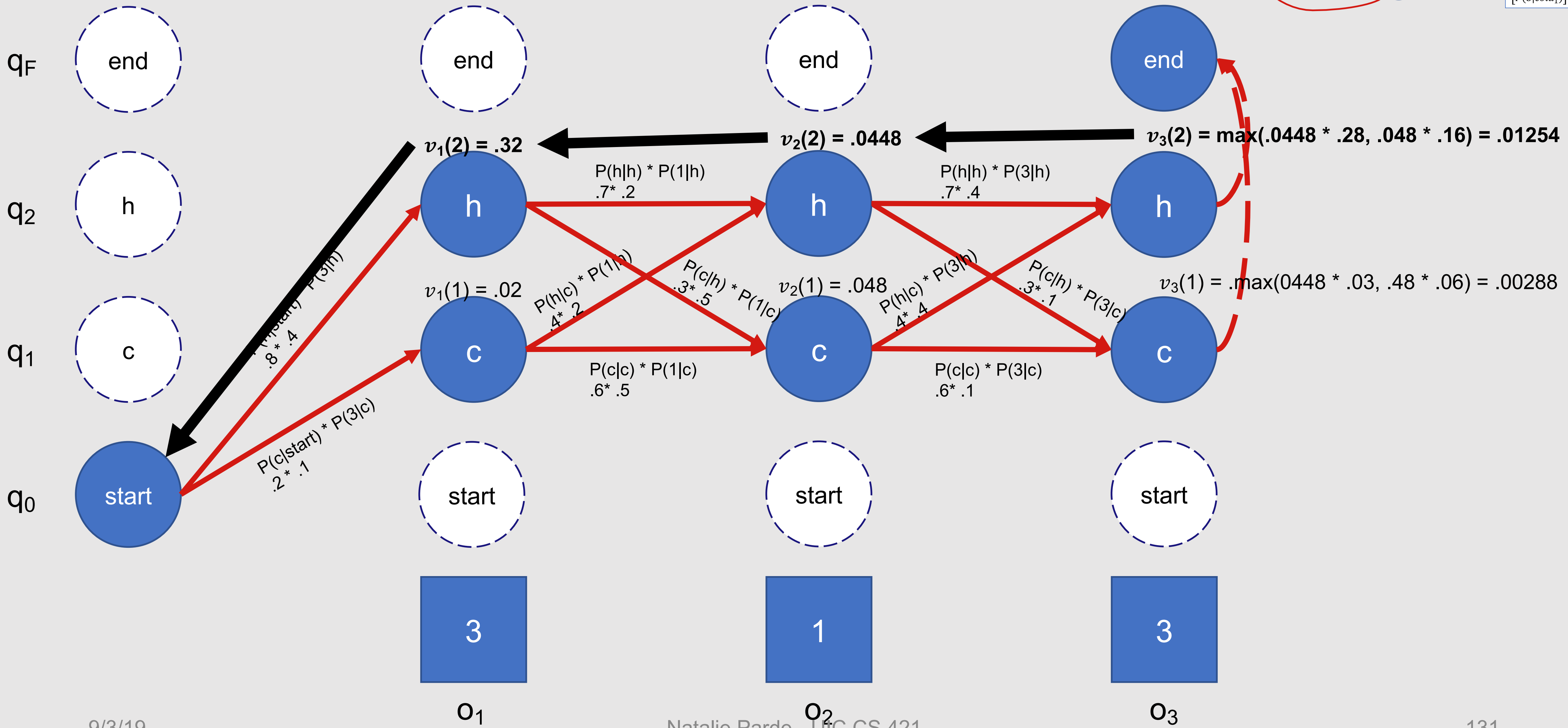
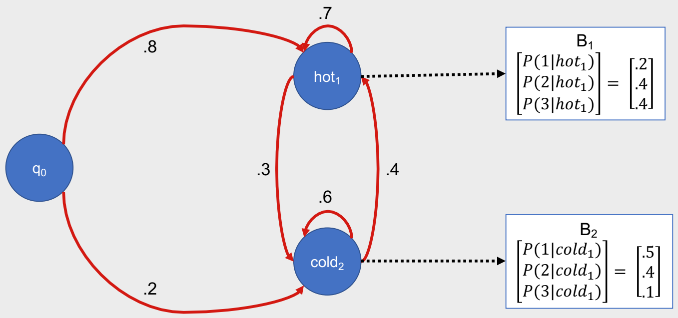




Learning

- If we have a set of observations, can we learn the parameters (transition probabilities and observation likelihoods) directly?

3 1 3
2 1 3
3 3 3
3 2 2
1 1 2



Forward-Backward Algorithm

- Special case of expectation-maximization (EM) algorithm
- Also known as the Baum-Welch algorithm
- Input:
  - Unlabeled sequence of observations, O
  - Vocabulary of hidden states, Q

- Example:
  - O = {3, 1, 3}
  - Q = {H, C}

How would this work with observable Markov models?

- Run the model on observation sequence O
- Since it's not hidden, we know which states we went through, and therefore which transitions and observations were used
- Given that information:
  - B = $\{b_j(o_t)\}$: Since every state can only generate one observation symbol, observation likelihoods are all 1.0
  - A = $\{a_{ij}\}$: $a_{ij} = \frac{C(i \rightarrow j)}{\sum_{q \in Q} C(i \rightarrow q)}$

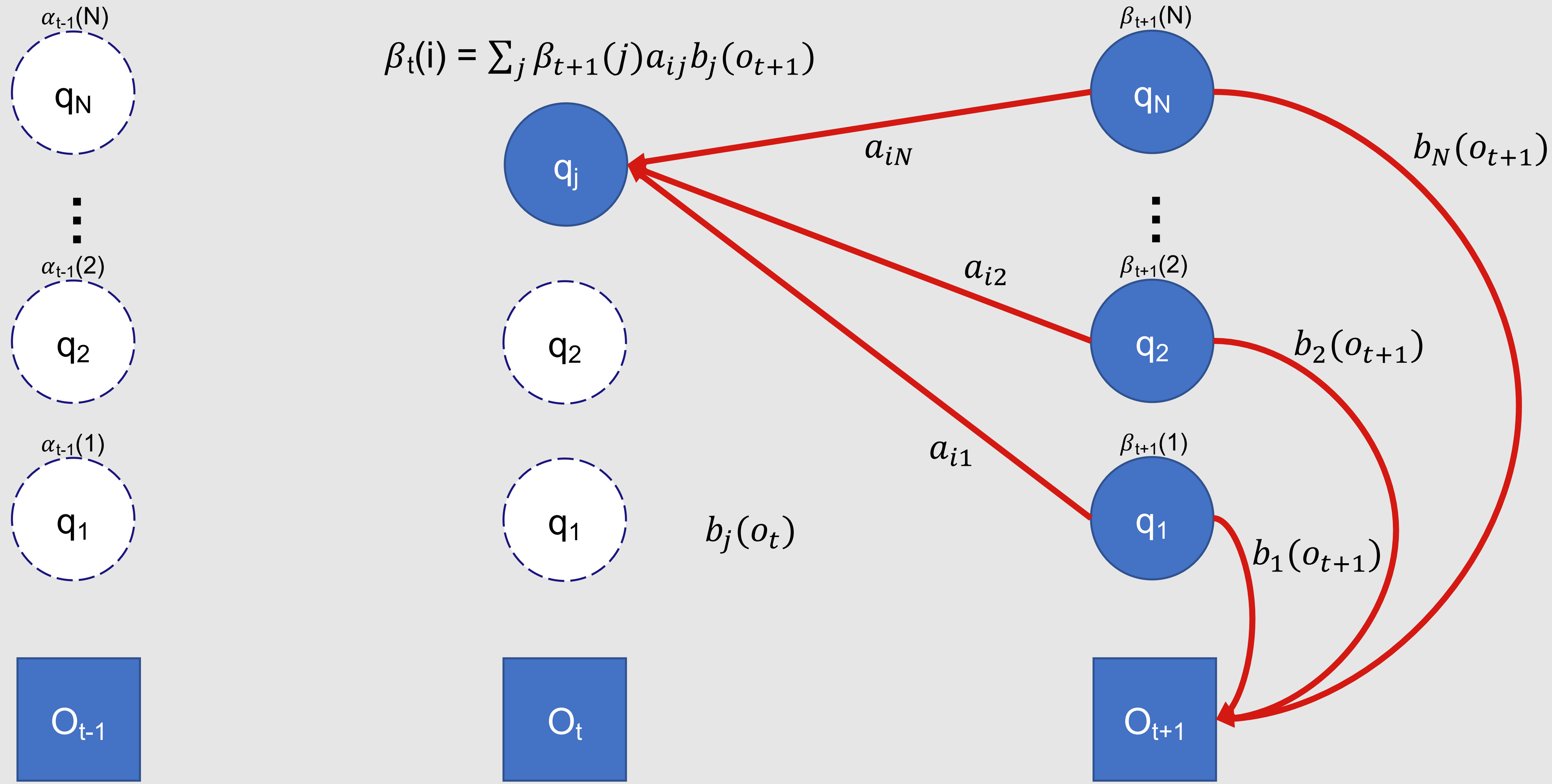
Extending this intuition to HMMs….

- We can't compute the counts directly from observed sequences
- Instead, we:
  - **Iteratively estimate the counts**
    - Start with base estimates for a_{ij} and b_j, and iteratively improve those estimates
  - **Get estimated probabilities** by:
    - Computing the forward probability for an observation
    - Dividing that probability mass among all the different paths that contributed to this forward probability

Backward Algorithm

- We define the backward probability as follows:
  - $\beta_t(i) = P(o_{t+1}, o_{t+2}, \ldots, o_T | q_t = i, \lambda)$
- This is the probability of generating partial observations from time *t*+1 until the end of the sequence, given that the HMM is in state *i* at time *t*

Backward Step



Re-Estimating a_{ij}

- We re-estimate a_{ij} as follows:
  - a_{ij}' = expected number of transitions from state *i* to state *j*, divided by expected number of transitions from state *I*

- More formally, we first define ξ as the probability of being in state *i* at time *t* and state *j* at time *t*+1, given the observation sequence and the HMM:
  - $\xi(i,j) = P(q_t = i, q_{t+1} = j|O,\lambda)$
- To compute ξ, we first define not-quite- ξ as a very similar probability with different conditioning of O:
  - not-quite-$\xi(i,j) = P(q_t = i, q_{t+1} = j, O|\lambda)$

Re-Estimating a_{ij}

- From not-quite- ξ, we can use Bayes rule ($P(X|Y,Z) = \frac{P(X,Y|Z)}{P(Y|Z)}$) to compute ξ:
  - $\xi_t(i,j) = \frac{\text{not}-\text{quite}-\xi_t(i,j)}{P(O|\lambda)}$
- This ends up being equivalent to:
  - $\xi_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}{\alpha_T(N)}$
- Finally, we can use this then to re-estimate a_{ij}:
  - $a'_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \sum_{j=1}^{N} \xi_t(i,j)}$

Re-Estimating Observation Likelihood

- We re-estimate b_j as follows:
  - $b_j'(v_k)$ = expected number of times in state *j* and observing vocabulary symbol v_k, divided by the expected number of times in state *j*
- Letting $\gamma_t(j)$ represent the probability of being in state *j* at time *t*, we can formally define the re-estimation as:
  - $b_j'(v_k) = \frac{\sum_{t=1\ s.t. o_t = v_k}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}$

Forward-Backward Algorithm

```
initialize A and B
iterate until convergence:
    # Expectation Step
    compute γ_t(j) for all t and j
    compute ξ_t(i,j) for all t, i, and j

    # Maximization Step
    recompute a_ij
    recompute b_j(v_k)
```

Summary: Hidden Markov Models

- HMMs are probabilistic generative models for sequences
- They make predictions based on underlying hidden states
- Three fundamental HMM problems include:
  - Computing the likelihood of a sequence of observations
  - Determining the best sequence of hidden states for an observed sequence
  - Learning HMM parameters given an observation sequence and a set of hidden states
- Observation likelihood can be computed using the forward algorithm
- Sequences of hidden states can be decoded using the Viterbi algorithm
- HMM parameters can be learned using the forward-backward algorithm